Direct Serendipity and Mixed Finite Elements on Polygons and Cuboidal Hexahedra

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This work was supported by
the U.S. National Science Foundation grant DMS-2111159

Objectives

Extend the theory of serendipity and mixed finite elements on n-dim cubes to polytopes

Serendipity finite elements. $\mathcal{S}_r$ on rectangle $\hat{E}_4$ in 2D:

- H^1-conforming
- Approximate to $\mathcal{O}(h^{r+1})$ with minimal # degrees of freedom (DoFs)

BDM mixed finite elements. BDM_r on rectangle $\hat{E}_4$ in 2D:

- $H(\mathrm{div})$-conforming
- Approximate velocity to $\mathcal{O}(h^{r+1})$ with minimal # of DoFs

Related by de Rham complex. (Arnold, Falk & Winther 2006)

$$\mathbb{R} \hookrightarrow \mathcal{S}_{r+1}(\hat{E}_4) \xrightarrow{\text{curl}} \mathrm{BDM}_r(\hat{E}_4) \xrightarrow{\text{div}} \mathbb{P}_{r-1}(\hat{E}_4) \longrightarrow 0$$

Problem. Lose accuracy when mapped to a quadrilateral E_4

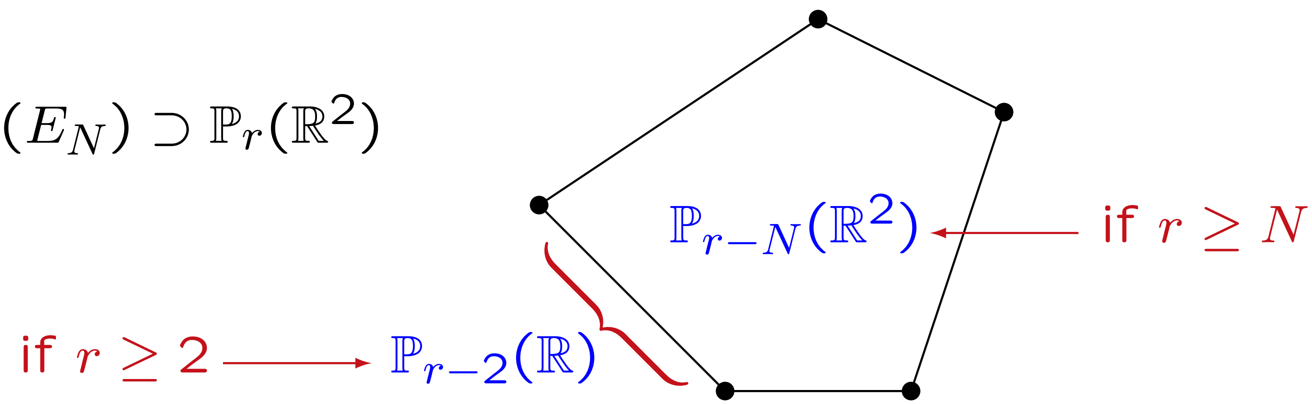
Goals. Define direct finite element spaces that

- Include polynomials $\mathbb{P}_r$ *directly* in the space (for approximation)
- Use minimal number of DoFs
- Apply to convex polygons E_N with N sides in 2D
- Extend to cuboidal hexahedra E in 3D

Direct Serendipity Finite Elements on Convex Polygons

Minimal # DoFs for H^1-Conformity on Polygons

Require $\mathcal{DS}_r(E_N) \supset \mathbb{P}_r(\mathbb{R}^2)$



DoFs required for H^1-Conformity ($N \geq 3$, $r \geq 1$)

Object	Object Count	DoFs per Object	Total DoFs
vertex	N	1	N
interior edge	N	$\dim \mathbb{P}_{r-2}(\mathbb{R})$	$N(r-1)$
interior cell	1	$\dim \mathbb{P}_{r-N}(\mathbb{R}^2)$	$\underbrace{\frac{1}{2}(r-N+2)(r-N+1)}_{\text{provided } r \geq N-2}$

The DoF counts imply

$$\mathcal{DS}_r(E_N) = \mathbb{P}_r(E_N) \oplus \mathbb{S}_r^{\mathcal{DS}}(E_N)$$

where

$$\mathbb{P}_r(E_N) \cap \mathbb{S}_r^{\mathcal{DS}}(E_N) = \emptyset$$

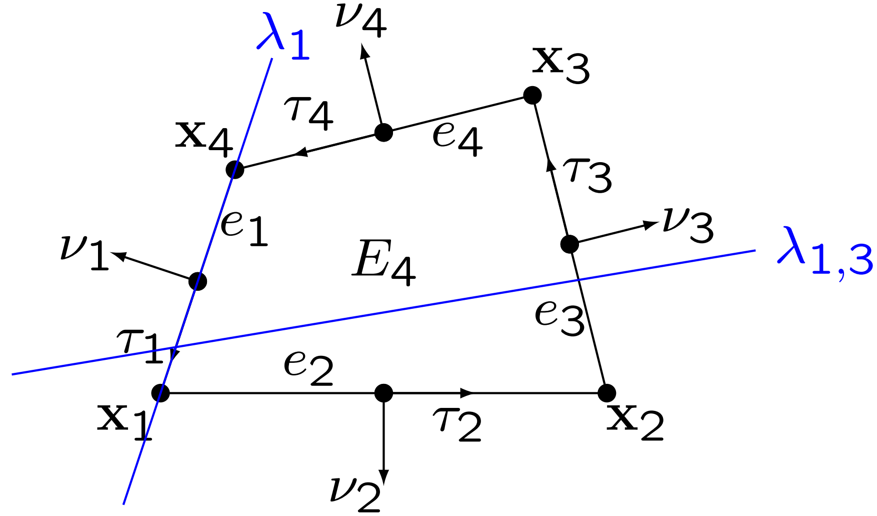
$$\dim \mathbb{S}_r^{\mathcal{DS}}(E_N) = \begin{cases} \frac{1}{2}N(N-3), & r \geq N-2 \\ Nr - \frac{1}{2}(r+2)(r+1) < \frac{1}{2}N(N-3), & r < N-2 \end{cases}$$

Counterintuitive observation. The case $r \geq N-2$ is easier!
For each of the N edges, there are $N-3$ nonadjacent edges. That is, the number of nonadjacent edge pairs is

$$\frac{1}{2}N(N-3) = \dim \mathbb{S}_r^{\mathcal{DS}}(E_N)$$

Supplemental basis functions are associated to pairs of edges!

Use CCW Ordering



Linear polynomial λ_i for edge e_i. Define

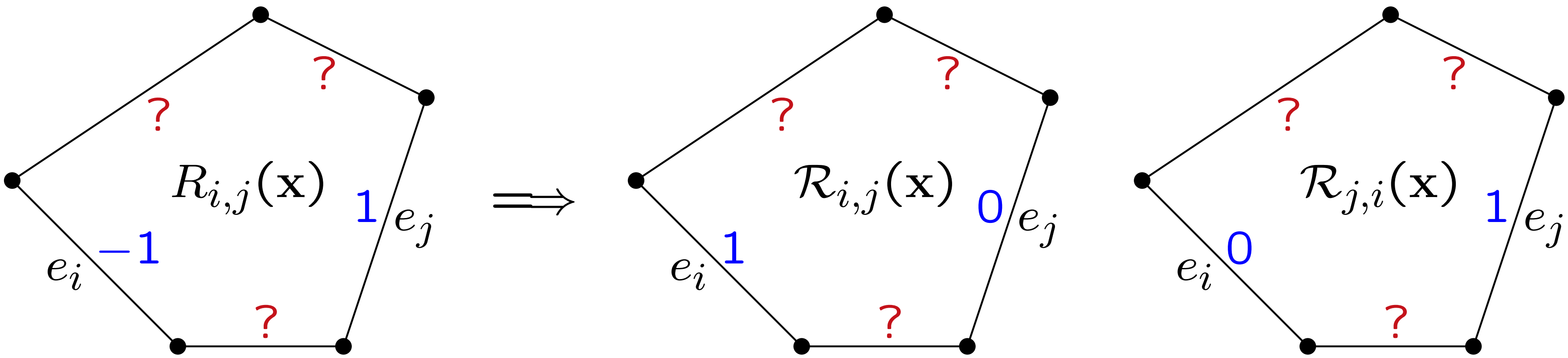
$$\lambda_i(\mathbf{x}) = -(\mathbf{x} - \mathbf{x}_i) \cdot \nu_i \quad \propto \quad \text{distance of } \mathbf{x} \text{ to the line through edge } e_i$$

$$\Longrightarrow \lambda_i\Big|_{e_i} = 0 \quad \text{(zero line contains } e_i\text{)}$$

$$\lambda_i > 0 \quad \text{on the interior of } E_N$$

These are not barycentric coordinates!

Linear polynomial $\lambda_{i,j}$ for edges e_i and e_j. Choose *any* linear polynomial $\lambda_{i,j}$ with zero line joining e_i and e_j.



$\mathcal{R}_{i,j}(\mathbf{x}) = \frac{1}{2}(1 - R_{i,j}(\mathbf{x}))$ $\quad$ $\mathcal{R}_{j,i}(\mathbf{x}) = \frac{1}{2}(1 + R_{i,j}(\mathbf{x}))$

Construction:

1. Rational functions

   When e_i and e_j are *not* adjacent and $i < j$, let

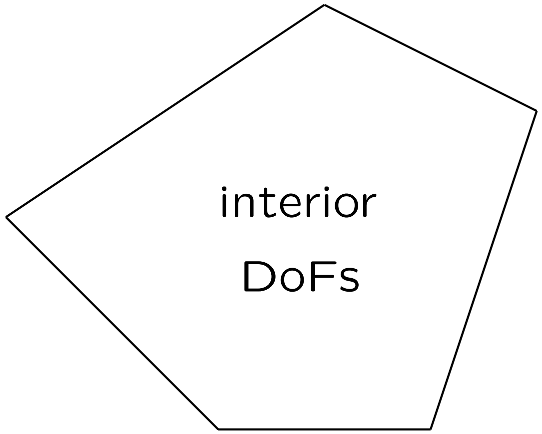
$$R_{i,j}(\mathbf{x}) = \frac{\lambda_i(\mathbf{x}) - \lambda_j(\mathbf{x})}{\lambda_i(\mathbf{x}) + \lambda_j(\mathbf{x})} \quad \Longrightarrow \quad \begin{cases} R_{i,j}(\mathbf{x})|_{e_i} = -1 \\ R_{i,j}(\mathbf{x})|_{e_j} = 1 \end{cases}$$

2. H^p piecewise polynomials of degree p on a sub-triangulation of E

Cell basis functions.

$$\omega_N(\mathbf{x}) = \prod_{i=1}^{N} \lambda_i(\mathbf{x})$$

vanishes on ∂E_N



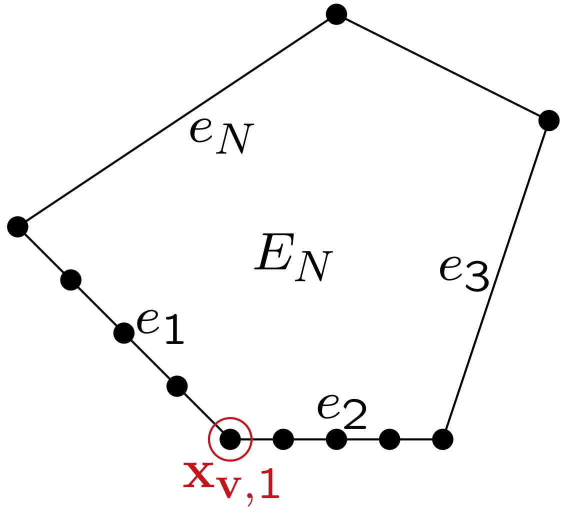
Bubble functions (if $r \geq N$)

$$\omega_N \mathbb{P}_{r-N}(E_N) \subset \mathbb{P}_r(E_N)$$

Vertex basis functions.

$$\tilde{\varphi}_{\mathbf{v},1} = \lambda_3 \lambda_4 \cdots \lambda_N \in \mathbb{P}_r(E_N)$$

Require $r \geq N-2$ here!



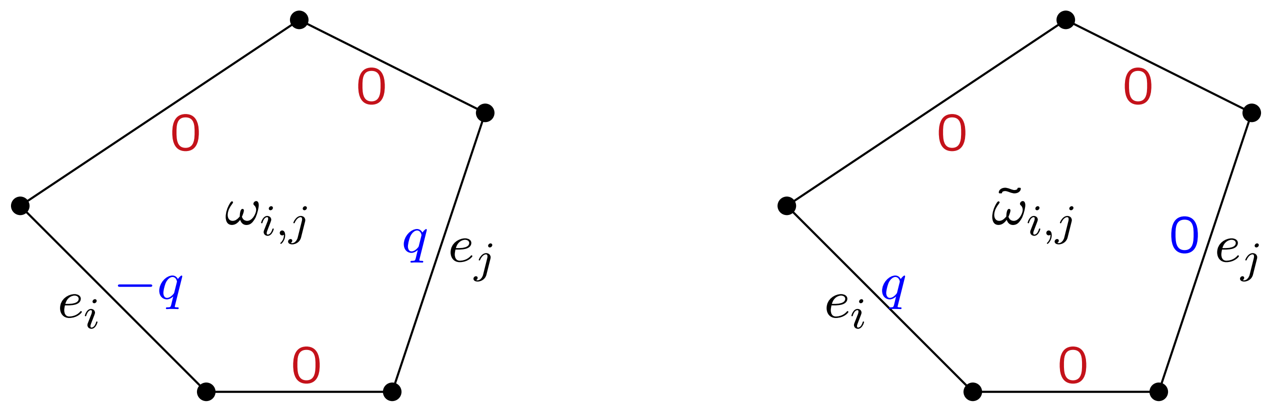
Remark. Can remove edge DoFs once define the edge basis functions

Supplements. $\dim \mathbb{S}_r^{\mathcal{DS}}(E_N) = \frac{1}{2}N(N-3)$ is the number of nonadjacent edge pairs. For such $i < j$,

$$\omega_{i,j} = \Big(\prod_{m \neq i,j} \lambda_m\Big)\lambda_{i,j}^{r-N+2} R_{i,j} \qquad \notin \mathbb{P}_r$$

Let

$$\tilde{\omega}_{i,j} = \Big(\prod_{m \neq i,j} \lambda_m\Big)\lambda_{i,j}^{r-N+2} \mathcal{R}_{i,j} \qquad \in \mathcal{DS}_r(E_N)$$



$$q = \Big(\prod_{m \neq i,j} \lambda_m\Big)\lambda_{i,j}^{r-N+2} \in \tilde{\mathbb{P}}_r(e_i)$$

General Case $\mathcal{DS}_r(E_N)$ Edge Basis, $r \geq N-2$ — 2

1. Isolate e_1. For $p \in \mathbb{P}_{r-N+1}$ (when $r \geq N-1$), let

$$\varphi_{e,1,1} = \underbrace{\Big(\prod_{m\neq 1} \lambda_m\Big) p}_{\text{Polynomials vanishing on other edges}} + \underbrace{\sum_{j\neq 1,2,N}}_{\text{Sum over nonadajacent edges}} b_j \underbrace{\tilde{\omega}_{1,j}}_{\text{Supplements that vanish on other edges}} \in \mathcal{DS}_r(E_N)$$

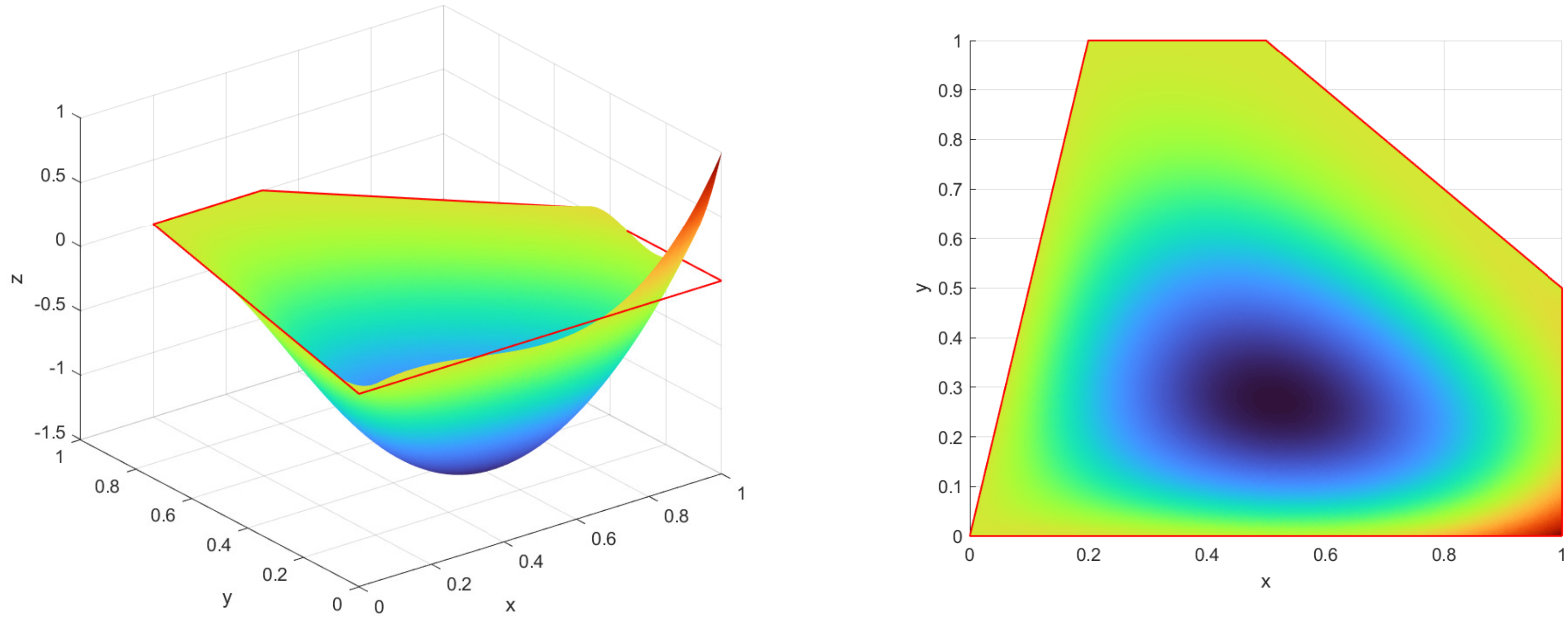
2. Clear other nodes. Select nontrivial b_j and a_ℓ of

$$p(\mathbf{x}) = a_0 + a_1(\mathbf{x}\cdot\tau_1) + \cdots + a_{r-N+1}(\mathbf{x}\cdot\tau_1)^{r-N+1}$$

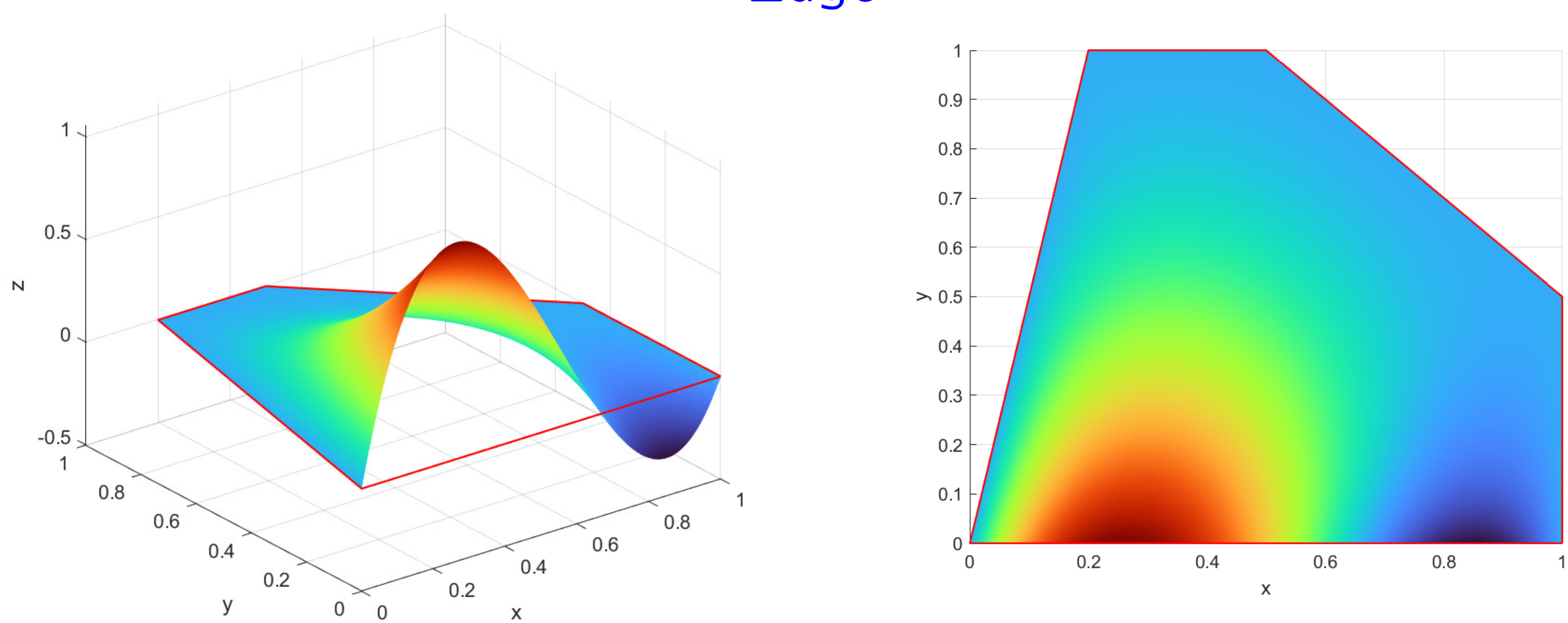
to kill the $r-2$ nodes $\mathbf{x}_{1,2}, \ldots, \mathbf{x}_{1,r-1}$.

- One can also set $\varphi_{e,1,1}(\mathbf{x}_{e,1,1}) = 1$ in this step.
- Theorem: The coefficients exist uniquely.
- The coefficients can be found by solving a small linear system.

Vertex



Edge



Final Remarks for $\mathcal{DS}_r(E_N)$ on Polygons

$r \geq N-2$ The finite element

$$\mathcal{DS}_r(E_N) = \mathbb{P}_r(E_N) \oplus \mathbb{S}_r^{\mathcal{DS}}(E_N)$$
$$\mathbb{S}_r^{\mathcal{DS}}(E_N) = \{\omega_{i,j} : i,j = 1,\ldots,N,\ e_i, e_j \text{ nonadjacent}\}$$
$$\omega_{i,j} = \Big(\prod_{m\neq i,j} \lambda_m\Big)\lambda_{i,j}^{r-N+2} R_{i,j}$$

(with nodal DoFs) is well defined (i.e., unisolvent).

Moreover, it has the minimal number of DoFs needed to

- contain $\mathbb{P}_r$
- be H^1 conforming

$r < N-2$

$$\mathcal{DS}_r(E_N) = \Big\{\varphi \in \mathcal{DS}_{N-2}(E_N) : \varphi|_{e_i} \in \mathbb{P}_r(e_i) \text{ for all edges } e_i \text{ of } E_N\Big\}$$

Theorem: $\mathcal{DS}_r(E_N) = \mathbb{P}_r(E_N) \oplus \mathbb{S}_r^{\mathcal{DS}}(E_N)$

Direct Serendipity Finite Elements on Cuboidal Hexahedra

DoFs ($r < 3$).

$$\#\text{DoFs} = \begin{cases} 8 = \dim \mathbb{P}_1(E) + 4, & \text{if } r = 1 \\ 20 = \dim \mathbb{P}_2(E) + 10, & \text{if } r = 2 \end{cases}$$

DoFs ($r \geq 3$).

Dimension	Object	Number	DoFs/Object	Total DoFs
0	vertex	8	1	8
1	edge	12	$\dim \mathbb{P}_{r-2}(\mathbb{R})$	$12(r-1)$
2	face	6	$\dim \mathbb{P}_{r-4}(\mathbb{R}^2)$	$3(r-2)(r-3)$, if $r \geq 2$
3	interior	1	$\dim \mathbb{P}_{r-6}(\mathbb{R}^3)$	$\frac{1}{6}(r-3)(r-4)(r-5)$, if $r \geq 3$

We must add 3(r+1) supplements to $\mathbb{P}_r$

Degrees of Freedom. Evaluation $\phi(\mathbf{v})$ at all the vertices $\mathbf{v}$, and for all the edges e and all the faces f,

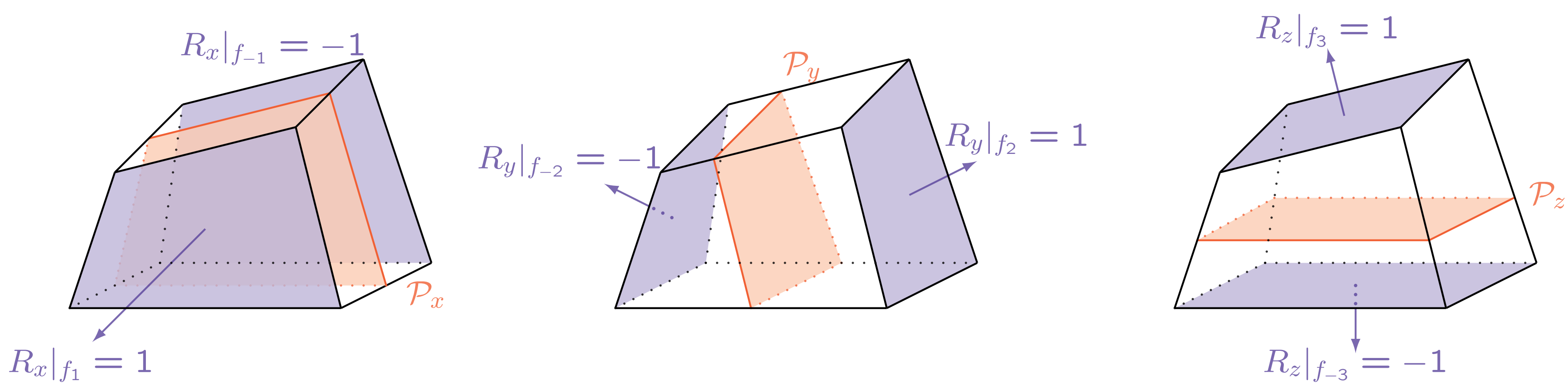
$$\int_e \phi\, q, \quad \forall q \in \mathbb{P}_{r-2}(e), \qquad \int_f \phi\, q, \quad \forall q \in \mathbb{P}_{r-4}(f), \qquad \int_E \phi\, q, \quad \forall q \in \mathbb{P}_{r-6}(E).$$

Linear Functions.

1. λ_n, for $n = \pm 1, \pm 2, \pm 3$:
$$\lambda_n(\mathbf{x}) = -(\mathbf{x} - \mathbf{x}_{f_n}) \cdot \nu_n$$
where $\mathbf{x}_{f_n} \in f_n$, and ν_n is the unit outer normal to face f_n.
2. For λ_x, λ_y, and λ_z: zero plane $\mathcal{P}_*$
   - $\mathcal{P}_x$ intersects $e_{\pm 2, \pm 3}$, $\mathcal{P}_y$ intersects $e_{\pm 1, \pm 3}$, $\mathcal{P}_z$ intersects $e_{\pm 1, \pm 2}$
   - $\mathcal{P}_x$, $\mathcal{P}_y$, and $\mathcal{P}_z$ do not coincide

Special maps.

$$R_x = \begin{cases} -1, & \text{on } f_{-1}, \\ 1, & \text{on } f_1, \end{cases} \quad R_y = \begin{cases} -1, & \text{on } f_{-2}, \\ 1, & \text{on } f_2, \end{cases} \quad R_z = \begin{cases} -1, & \text{on } f_{-3}, \\ 1, & \text{on } f_3. \end{cases}$$



$r = 1, 2$: Construct $\mathcal{DS}_r(E) \subset \mathcal{DS}_3(E)$.

$r \geq 3$:

1. Vertex basis functions

$$\lambda_{-i}\lambda_{-j}\lambda_{-k} \in \mathbb{P}_r$$

2. Cell basis functions

$$\lambda_{-1}\lambda_1\lambda_{-2}\lambda_2\lambda_{-3}\lambda_3\,\mathbb{P}_{r-6} \subset \mathbb{P}_r$$

3. Face basis functions
   - For face f_m: $\left(\prod_{n \neq m} \lambda_n\right)\mathbb{P}_{r-5} \subset \mathbb{P}_r$
   - $3(r-1)$ simple supplements needed (like in 2D)
4. Edge basis functions
   - For edge $e_{m,l}$: $\left(\prod_{n \neq m,l} \lambda_n\right)\mathbb{P}_{r-4} \subset \mathbb{P}_r$
   - 9 simple supplements needed (like in 2D)
   - 3 complex supplements needed (unlike in 2D)

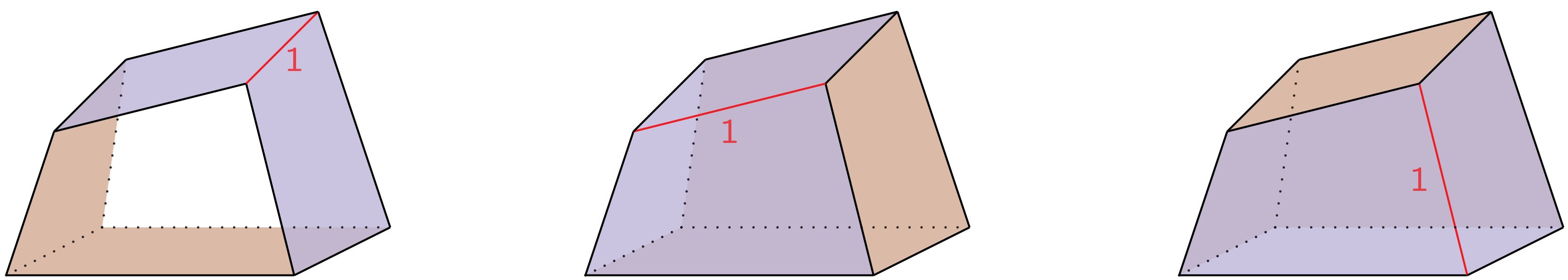
Unisolvence. Unisolvence lies in the construction of shape functions.

Idea. The restriction of the supplements to each face f should fall into $\mathcal{DS}_r^{(2)}(f)$ defined on quadrilateral f (Arbogast, Tao, & Wang 2022)

Complex Supplemental functions.

$$\lambda_{-1}\lambda_1\lambda_x^{r-3}\psi_x, \qquad \lambda_{-2}\lambda_2\lambda_y^{r-3}\psi_y, \qquad \lambda_{-3}\lambda_3\lambda_z^{r-3}\psi_z$$

Requirements of ψ_x, ψ_y, and ψ_z on faces.



$$\psi_x|_{f_{-2}\cup f_{-3}} = 0 \qquad \psi_y|_{f_{-1}\cup f_{-3}} = 0 \qquad \psi_z|_{f_{-1}\cup f_{-2}} = 0$$

$$\psi_x|_{e_{2,3}} = 1 \qquad \psi_y|_{e_{1,3}} = 1 \qquad \psi_z|_{e_{1,2}} = 1$$

$$\psi_x|_{f_2} \in \mathbb{P}_1 \oplus \{\lambda_x R_z\} \qquad \psi_y|_{f_1} \in \mathbb{P}_1 \oplus \{\lambda_y R_z\} \qquad \psi_z|_{f_1} \in \mathbb{P}_1 \oplus \{\lambda_z R_y\}$$

$$\psi_x|_{f_3} \in \mathbb{P}_1 \oplus \{\lambda_x R_y\} \qquad \psi_y|_{f_3} \in \mathbb{P}_1 \oplus \{\lambda_y R_x\} \qquad \psi_z|_{f_2} \in \mathbb{P}_1 \oplus \{\lambda_z R_x\}$$

Conformity. Match $\mathcal{DS}_r^{(2)}(f)$ for neighbor elements sharing f.

Direct Mixed Finite Elements on Convex Polygons

De Rham Complex. The image of one map is the kernel of the next.

$$\mathbb{R} \hookrightarrow \mathcal{DS}_{r+1}(E_N) \xrightarrow{\text{curl}} \mathbf{V}_r^s(E_N) \xrightarrow{\text{div}} \mathbb{P}_s(E_N) \longrightarrow 0$$

Recall

- $\mathbb{P}_r^2 = \text{curl}\,\mathbb{P}_{r+1} \oplus \mathbf{x}\mathbb{P}_{r-1}$
- $\nabla\cdot : \mathbf{x}\mathbb{P}_s \to \mathbb{P}_s$ is one-to-one and onto

Decomposition into direct finite elements.

Reduced $H(\text{div})$-approximating: $s = r - 1$, $r = 1, 2, \ldots$

$$\begin{aligned}\mathbf{V}_r^{r-1}(E_N) &= \text{curl}\,\mathcal{DS}_{r+1}(E_N) \oplus \mathbf{x}\mathbb{P}_{r-1} \\ &= \mathbb{P}_r^2(E_N) \oplus \mathbb{S}_r^{\mathbf{V}}(E_N)\end{aligned}$$

Full $H(\text{div})$-approximating: $s = r$, $r = 0, 1, \ldots$

$$\begin{aligned}\mathbf{V}_r^{r}(E_N) &= \text{curl}\,\mathcal{DS}_{r+1}(E_N) \oplus \mathbf{x}\mathbb{P}_{r} \\ &= \mathbb{P}_r^2(E_N) \oplus \mathbf{x}\underbrace{\tilde{\mathbb{P}}_r}_{\text{homogeneous polynomials}} \oplus \mathbb{S}_r^{\mathbf{V}}(E_N)\end{aligned}$$

The supplemental (vector valued) functions are

$$\mathbb{S}_r^{\mathbf{V}}(E_N) = \text{curl}\,\mathbb{S}_{r+1}^{\mathcal{DS}}(E_N)$$

Approximation Properties

Approximation Properties of $\mathcal{DS}_r^s$ (2D and 3D)

Generalized Scott-Zhang Interpolation Operator. For $\mathcal{DS}_r = \text{span}\{\phi_j\}$,

$$\mathcal{I}_h^r : W_p^l(\Omega) \to \mathcal{DS}_r(\Omega)$$

Theorem (Optimal approx.). Assume $\mathcal{T}_h$ is uniformly shape regular and

$$1 \leq p \leq \infty \quad \text{and} \quad l > 1/p \ (l \geq 1 \text{ if } p = 1)$$

Then there exists $C > 0$, such that

$$||v - \mathcal{I}_h^r v||_{W_p^m(E)} \leq C\, h_E^{l-m} |v|_{W_p^l(E)}, \quad 0 \leq m \leq l \leq r+1$$

Moreover,

$$\Big(\sum_{E \in \mathcal{T}_h} ||v - \mathcal{I}_h^r v||_{W_p^m(E)}^p \Big)^{1/p} \leq C\, h^{l-m}\, |v|_{W_p^l(\Omega)}, \quad 0 \leq m \leq l \leq r+1$$

Approximation Properties of V_r^s (2D)

Generalized Raviart-Thomas projection operator.

$$\pi : H(\mathrm{div};\Omega) \cap (L^{2+\epsilon}(\Omega))^2 \to \mathbf{V}_r^s, \quad s = r-1, r \ (s \geq 0)$$

Theorem (Optimal approximation). For $s = r-1, r$ $(s \geq 0)$, define

$$\begin{aligned} \|\mathbf{v} - \pi\mathbf{v}\|_{0,\Omega} &\leq C\,\|\mathbf{v}\|_{k,\Omega}\,h^k & k &= 1,\dots,r+1 \\ \|\nabla\cdot(\mathbf{v} - \pi\mathbf{v})\|_{0,\Omega} &\leq C\,\|\nabla\cdot\mathbf{v}\|_{k,\Omega}\,h^k & k &= 0,1,\dots,s+1 \\ \|p - \mathcal{P}_{W_s}p\|_{0,\Omega} &\leq C\,\|p\|_{k,\Omega}\,h^k & k &= 0,1,\dots,s+1 \end{aligned}$$

Moreover, the discrete inf-sup condition holds for some $\gamma > 0$:

$$\sup_{\mathbf{v}_h \in \mathbf{V}_r^s} \frac{(w_h, \nabla\cdot\mathbf{v}_h)}{\|\mathbf{v}_h\|_{H(\mathrm{div})}} \geq \gamma\,\|w_h\|_{0,\Omega}, \quad \forall w_h \in W_s = \mathbb{P}_s$$

Numerical Tests

Convergence for $\mathcal{DS}_r$ on Polygons

Test problem.

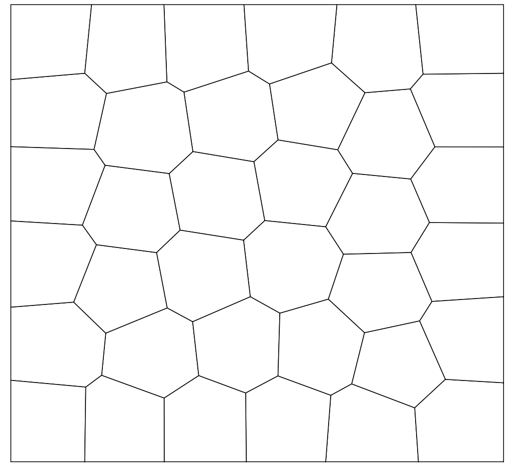
$$-\Delta u = f$$

True solution.

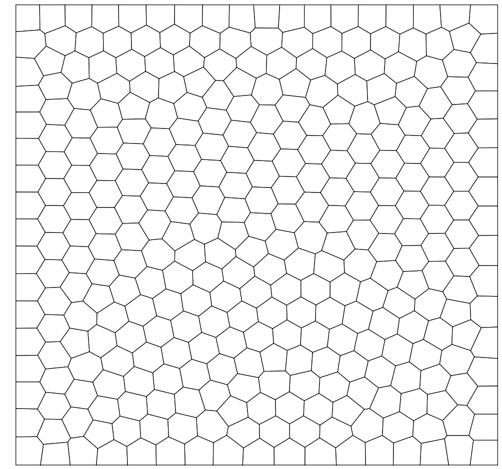
$$u(x, y) = \sin(\pi x)\sin(\pi y)$$

Randomly generated mesh.

Piecewise polynomial supplements.



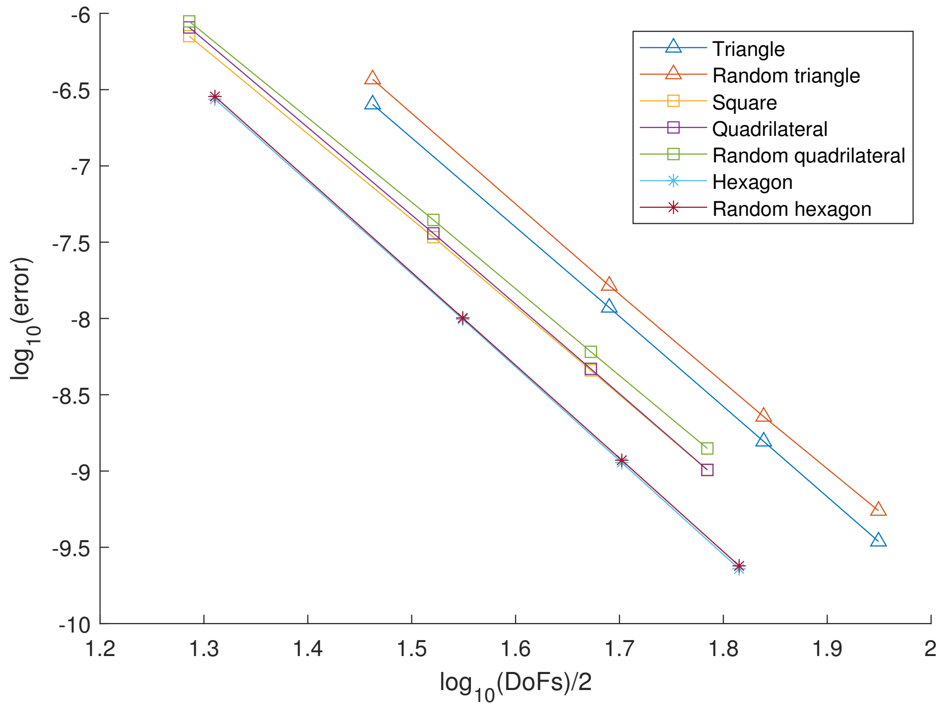
$n = 6$



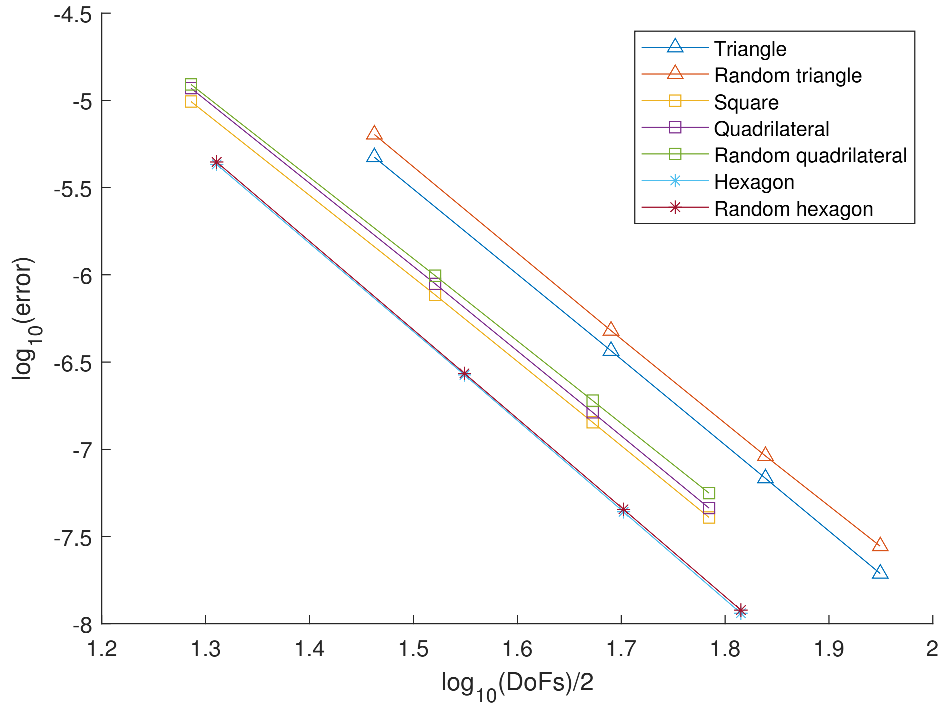
$n = 18$

n	$r = 2$		$r = 3$		$r = 4$		$r = 5$	
				L^2-norm				
10	2.144e-04	3.50	1.031e-05	4.35	3.972e-07	6.11	1.730e-08	6.61
14	7.165e-05	3.21	2.518e-06	4.13	6.622e-08	5.24	1.964e-09	6.37
18	3.409e-05	2.92	8.964e-07	4.05	1.823e-08	5.06	4.134e-10	6.12
22	1.841e-05	3.46	4.045e-07	4.48	6.239e-09	6.03	1.243e-10	6.76
		$\mathcal{O}(h^3)$		$\mathcal{O}(h^4)$		$\mathcal{O}(h^5)$		$\mathcal{O}(h^6)$
				H^1-seminorm				
10	3.552e-03	2.36	2.390e-04	3.11	1.027e-05	4.91	4.394e-07	5.57
14	1.660e-03	2.23	8.343e-05	3.08	2.439e-06	4.21	7.098e-08	5.34
18	1.013e-03	1.94	3.783e-05	3.10	8.785e-07	4.01	1.981e-08	5.01
22	6.696e-04	2.33	2.114e-05	3.27	3.689e-07	4.88	7.233e-09	5.66
		$\mathcal{O}(h^2)$		$\mathcal{O}(h^3)$		$\mathcal{O}(h^4)$		$\mathcal{O}(h^5)$

DoF Efficiency Study for $\mathcal{DS}_r$ on Polygons



L^2-norm



H^1-seminorm

Comparison of triangular, quadrilateral, and hexagonal meshes with $n = 6, 10, 14, 18, 22$ for $r = 5$.

Convergence for V_r^s on Polygons

Errors and convergence rates for direct mixed spaces

n	$\|\|p - p_h\|\|$ error	rate	$\|\mathbf{u} - \mathbf{u}_h\|$ error	rate	$\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|$ error	rate
$r = s = 1$, full H(div)-approximation						
10	8.635e-03	2.23	1.892e-03	2.67	8.634e-03	2.23
14	4.308e-03	2.04	8.562e-04	2.32	4.308e-03	2.03
18	2.614e-03	1.96	4.895e-04	2.19	2.614e-03	1.96
22	1.715e-03	2.37	3.090e-04	2.59	1.715e-03	2.37
		$\mathcal{O}(h^2)$		$\mathcal{O}(h^2)$		$\mathcal{O}(h^2)$
$r = 1$, $s = 0$, reduced H(div)-approximation						
10	1.290e-01	1.24	1.770e-02	2.29	1.260e-01	1.15
14	9.109e-02	1.02	8.997e-03	1.98	9.001e-02	0.98
18	7.039e-02	1.01	5.428e-03	1.98	6.988e-02	0.99
22	5.736e-02	1.15	3.620e-03	2.28	5.708e-02	1.14
		$\mathcal{O}(h^1)$		$\mathcal{O}(h^2)$		$\mathcal{O}(h^1)$
$r = s = 2$, full H(div)-approximation						
10	3.881e-04	3.38	6.546e-05	3.74	3.881e-04	3.38
14	1.384e-04	3.02	1.945e-05	3.55	1.384e-04	3.02
18	6.515e-05	2.96	8.818e-06	3.10	6.515e-05	2.96
22	3.507e-05	3.48	4.349e-06	3.97	3.507e-05	3.48
		$\mathcal{O}(h^3)$		$\mathcal{O}(h^3)$		$\mathcal{O}(h^3)$
$r = 2$, $s = 1$, reduced H(div)-approximation						
10	8.635e-03	2.23	5.013e-04	3.24	8.634e-03	2.23
14	4.308e-03	2.04	1.785e-04	3.02	4.308e-03	2.03
18	2.614e-03	1.96	8.492e-05	2.92	2.614e-03	1.96
22	1.715e-03	2.37	4.637e-05	3.40	1.715e-03	2.37
		$\mathcal{O}(h^2)$		$\mathcal{O}(h^3)$		$\mathcal{O}(h^2)$

Convergence for $\mathcal{DS}_r$ on Cuboidal Hexahedra

Test problem.

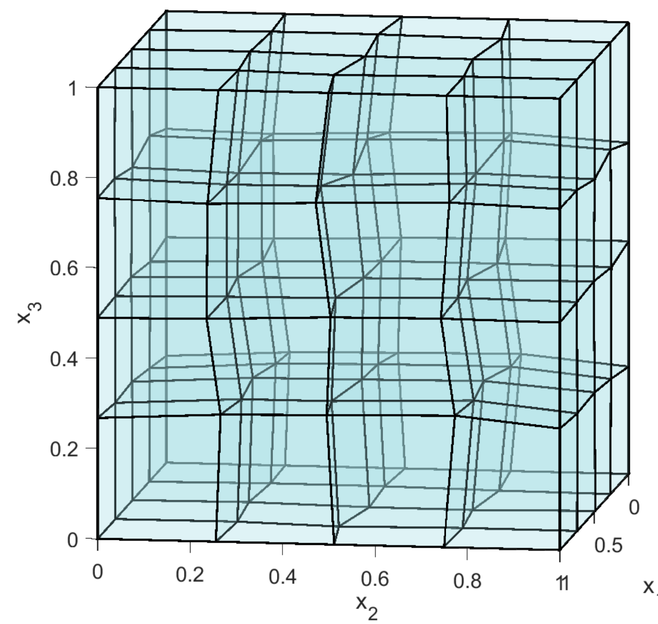
$$-\Delta u = f$$

True solution.

$$u(x,y) = \sin(\pi x)\sin(\pi y)\sin(\pi z)$$

Randomly generated mesh.

Smooth supplements.



n	$r=1$		$r=2$		$r=3$		$r=4$	
				L^2-norm				
4	7.508e-02	—	5.109e-03	—	1.317e-03	—	1.770e-04	—
8	1.797e-02	2.21	6.252e-04	3.24	7.446e-05	4.43	5.327e-06	5.40
12	7.626e-03	2.12	1.885e-04	2.96	1.342e-05	4.24	6.956e-07	5.03
16	4.349e-03	2.30	8.192e-05	3.41	4.406e-06	4.56	1.720e-07	5.73
		$\mathcal{O}(h^2)$		$\mathcal{O}(h^3)$		$\mathcal{O}(h^4)$		$\mathcal{O}(h^5)$
				H^1-seminorm				
4	2.410e-01	—	2.571e-02	—	9.244e-03	—	1.263e-03	—
8	1.181e-01	1.10	6.006e-03	2.24	1.092e-03	3.29	7.755e-05	4.30
12	7.697e-02	1.06	2.677e-03	2.00	3.031e-04	3.17	1.533e-05	4.01
16	5.788e-02	1.17	1.527e-03	2.30	1.307e-04	3.45	5.002e-06	4.59
		$\mathcal{O}(h^1)$		$\mathcal{O}(h^2)$		$\mathcal{O}(h^3)$		$\mathcal{O}(h^4)$

Summary and Conclusions

1. Direct spaces (polynomials plus supplements) offer advantages
   - Quadrilaterals and hexahedra: no accuracy loss due to reference element mapping.
   - Polygons: require no reference element.
2. Direct serendipity finite elements developed for convex polygons and cuboidal hexahedra.
   - H^1-conforming and fully constructive. Keys to the construction:
     - Lower order spaces: subspace of a higher order space
     - Higher order spaces:
       - Convex polygons: edge basis functions require supplements.
       - Cuboidal hexahedra: face and edge basis functions require supplements, $\mathcal{DS}_r(E)|_f$ coincides $\mathcal{DS}_r^{(2)}(f)$.
   - Minimal DoFs and approximate optimally on shape regular meshes.
   - Polygons with more edges give less error.

3. Direct mixed finite elements developed for convex polygons.
   - $H(\text{div})$-conforming direct and fully constructive.
   - Arise from a de Rham complex using FEEC.
   - Full and reduced $H(\text{div})$-approximating spaces.
   - Minimal DoFs and approximate optimally on shape regular meshes.
4. Future work
   - Applications.
   - De Rham complex and direct mixed finite elements for cuboidal hexahedra.
   - Extension to more general 3D polytopes.

References

1. T. Arbogast and Chuning Wang: Direct serendipity finite elements on cuboidal hexahedra, in preparation.
2. T. Arbogast and Chuning Wang: Direct serendipity and mixed finite elements on convex polygons, Numer. Algorithms, 92:1451–1483, 2023.
3. T. Arbogast, Zhen Tao, and Chuning Wang: Direct serendipity and mixed finite elements on convex quadrilaterals, Numer. Math., 150:929–974, 2022.
4. T. Arbogast and Zhen Tao: A direct mixed-enriched Galerkin method on quadrilaterals for two-phase Darcy flow. Computational Geosci., 23(5):1141–1160, 2019.
5. T. Arbogast and Zhen Tao: Construction of H(div)-conforming mixed finite elements on cuboidal hexahedra. Numer. Math., 142(1):1–32, 2019.
6. T. Arbogast and M. R. Correa: Two families of H(div) mixed finite elements on quadrilaterals of minimal dimension. SIAM J. Numer. Anal., 54(6):3332–3356, 2016.