Direct Serendipity and Mixed Finite Elements on Quadrilaterals for Flow and Transport in Porous Media

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Motivation

Accurately solve flow and transport in porous media

Requirements.

- Local mass conservation
- Meshes of quadrilaterals to follow subsurface layering
- Accuracy with computational efficiency

Strategy.

- Transport: Use the enriched Galerkin method
  - Continuous finite elements enriched with discontinuous constants
  - Discontinuous Galerkin bilinear form
  - Tensor product elements have excess DoFs: use serendipity spaces
- Flow: Use the mixed finite element method

Problem.

- Serendipity and mixed elements lose accuracy on quadrilaterals!

Extend the theory of serendipity and mixed elements on rectangles to quadrilaterals, and apply to flow and transport

Serendipity finite elements. $\mathcal{S}_r$ on rectangle $\hat{E}$:

- H^1-conforming
- Approximate to $\mathcal{O}(h^{r+1})$ with minimal # degrees of freedom (DoFs)

BDM mixed finite elements. BDM_r on rectangle $\hat{E}$:

- $H(\mathrm{div})$-conforming
- Approximate velocity to $\mathcal{O}(h^{r+1})$ with minimal # of DoFs

Related by de Rham complex.

$$\mathbb{R} \hookrightarrow \mathcal{S}_{r+1}(\hat{E}) \xrightarrow{\text{curl}} \mathrm{BDM}_r(\hat{E}) \xrightarrow{\text{div}} \mathbb{P}_{r-1}(\hat{E}) \longrightarrow 0$$

Problem. Lose accuracy when mapped to a quadrilateral E

Objective. Define direct finite element spaces that

- Include $\mathbb{P}_r(E)$ *directly* in the space (for approximation)
- Use minimal number of degrees of freedom

1. Direct Serendipity Spaces

$$\mathcal{DS}_r(E) = \mathbb{P}_r(E) \oplus \mathbb{S}_r^{\mathcal{DS}}(E)$$

Polynomials plus supplements

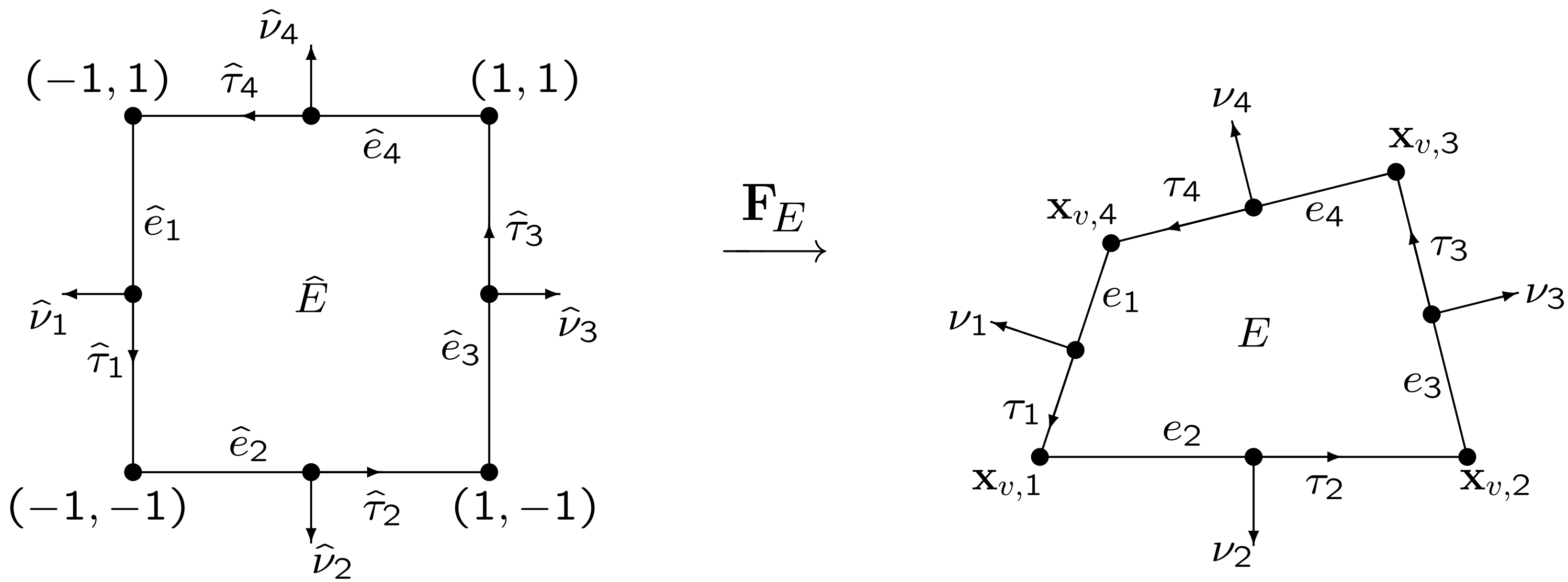
Lowest Order ($r = 1$)

(Arnold, Boffi & Falk 2002)

The usual 4 node serendipity space is defined by mapping from a reference rectangle $\hat{E}$ to E

$$\begin{aligned}\mathcal{S}_1(E) &= \text{span}\{F_E^0(1), F_E^0(\hat{x}), F_E^0(\hat{y}), F_E^0(\hat{x}\hat{y})\} \\ &= \text{span}\{1, x, y, F_E^0(\hat{x}\hat{y})\} \\ &= \mathbb{P}_1(E) \oplus \text{span}\{F_E^0(\hat{x}\hat{y})\} \\ &= \mathcal{DS}_1(E)\end{aligned}$$

The lowest order classic serendipity space naturally has the form of a direct serendipity space, so it is accurate



Bilinear map. $\mathbf{F}_E : \hat{E} \to E$

Linear polynomials.

$$\lambda_i(\mathbf{x}) = -(\mathbf{x} - \mathbf{x}_i) \cdot \nu_i \quad \propto \quad \text{distance of } \mathbf{x} \text{ to edge } e_i$$

$$\Longrightarrow \lambda_i\Big|_{e_i} = 0$$

Space of polynomials.

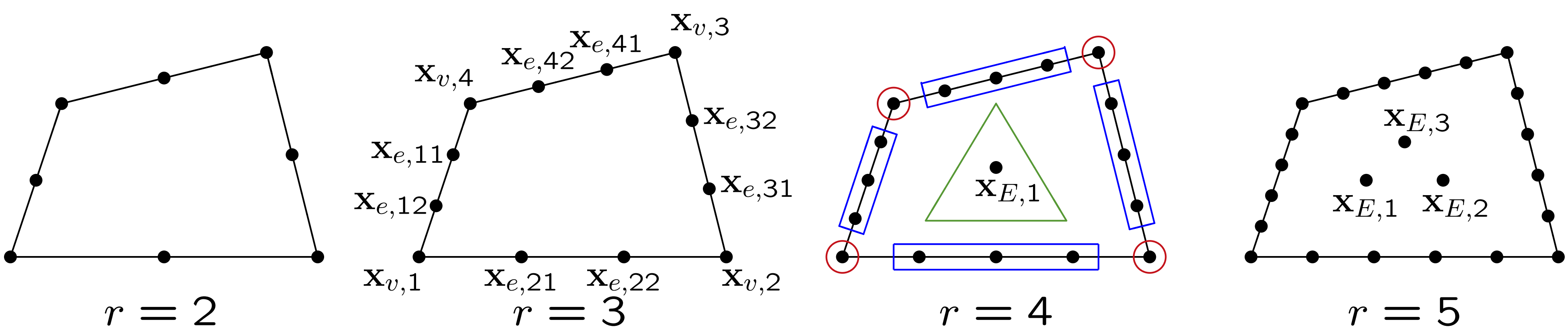
$$\dim \mathbb{P}_r(\Omega) = \binom{r+d}{d} = \frac{(r+d)!}{r!\,d!}, \qquad d = 0, 1, 2$$

Geometric decomposition and degrees of freedom (DoFs).

Dimension	Object	Number	DoFs/Object	Total DoFs
0	vertex	4	1	4
1	edge	4	$\dim \mathbb{P}_{r-2}(e)$	$4(r-1)$
2	cell	1	$\dim \mathbb{P}_{r-4}(E)$	$\frac{1}{2}(r-2)(r-3)$

total # DoFs $= \dim \mathbb{P}_r + 2 \implies$ We must add 2 supplements to $\mathbb{P}_r$

Nodal points for the DoFs.



$$\mathcal{DS}_r(E) = \mathbb{P}_r(E) \oplus \mathbb{S}_r^{\mathcal{DS}}(E)$$

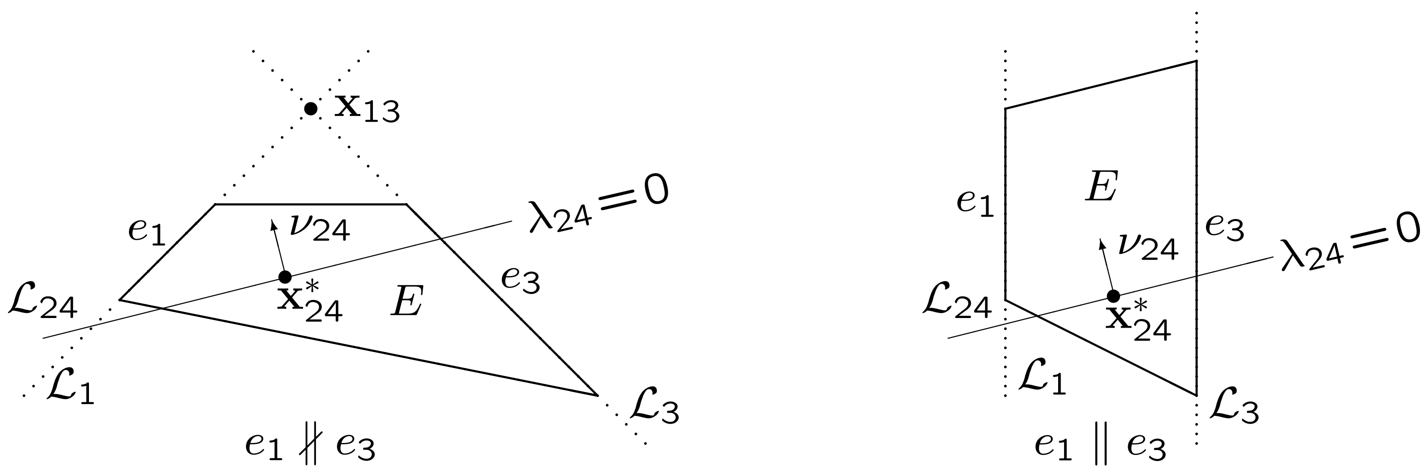
We get a family of direct serendipity elements for

$$\mathbb{S}_r^{\mathcal{DS}}(E) = \text{span}\{\lambda_2\lambda_4\lambda_{24}^{r-2}R_{13}, \lambda_1\lambda_3\lambda_{13}^{r-2}R_{24}\}$$

Choices.

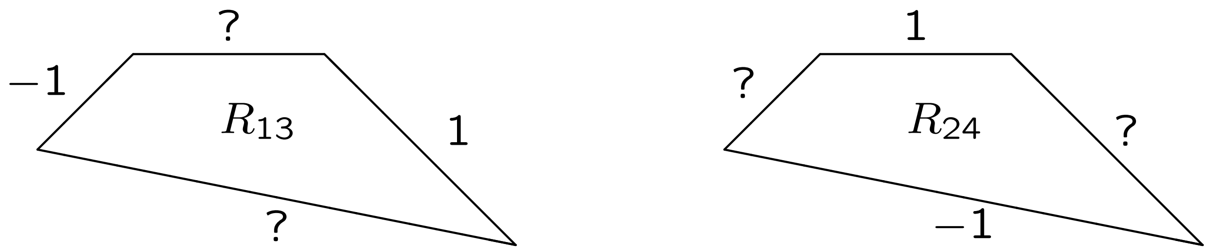
1. Linear functions λ_{24} and λ_{13}

$$\lambda_{24}(\mathbf{x}) = -(\mathbf{x} - \mathbf{x}_{24}^*) \cdot \nu_{24} \quad \text{and} \quad \lambda_{13}(\mathbf{x}) = -(\mathbf{x} - \mathbf{x}_{13}^*) \cdot \nu_{13}$$



Example: $\lambda_{24}^{\text{simple}} = \dfrac{\lambda_2 - \lambda_4}{\|\nu_2 - \nu_4\|}$ and $\lambda_{13}^{\text{simple}} = \dfrac{\lambda_1 - \lambda_3}{\|\nu_1 - \nu_3\|}$

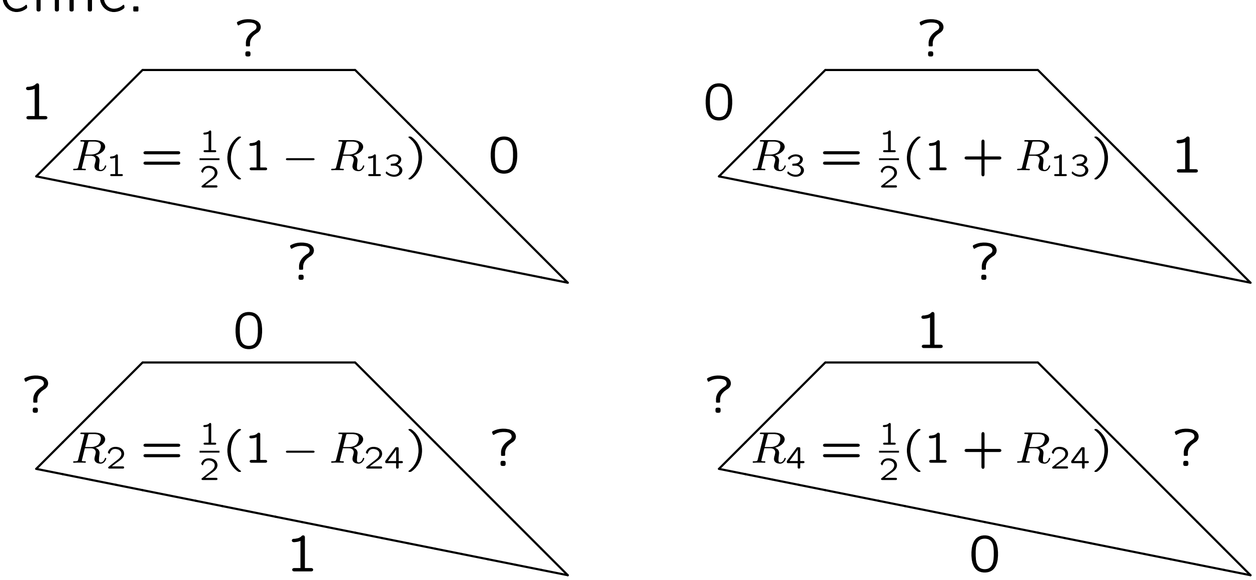
2. The functions R_{13} and R_{24} are defined to satisfy the properties



Example:

$$R_{13}^{\text{simple}}(\mathbf{x}) = \frac{\lambda_1(\mathbf{x}) - \lambda_3(\mathbf{x})}{\lambda_1(\mathbf{x}) + \lambda_3(\mathbf{x})} \quad \text{and} \quad R_{24}^{\text{simple}}(\mathbf{x}) = \frac{\lambda_2(\mathbf{x}) - \lambda_4(\mathbf{x})}{\lambda_2(\mathbf{x}) + \lambda_4(\mathbf{x})}$$

Further define:



Interior Nodal Basis Functions ($r \geq 4$)

Shape functions. $\dim \mathbb{P}_{r-4} = \frac{1}{2}(r-2)(r-3)$ shape functions

$$\lambda_1\lambda_2\lambda_3\lambda_4\mathbb{P}_{r-4}$$

Nodal basis functions. Let $\{\phi_{E,i}\} \subset \mathbb{P}_{r-4}$ be a nodal basis for the cell nodes $\{\mathbf{x}_{E,i}\}$, where $i = 1, \ldots, \dim \mathbb{P}_{r-4}$

$$\varphi_{E,i}(\mathbf{x}) = \frac{[\lambda_1\lambda_2\lambda_3\lambda_4](\mathbf{x})\phi_{E,i}(\mathbf{x})}{[\lambda_1\lambda_2\lambda_3\lambda_4](\mathbf{x}_{E,i})}, \quad i = 1, \ldots, \dim \mathbb{P}_{r-4}$$

Edge Nodal Basis Functions

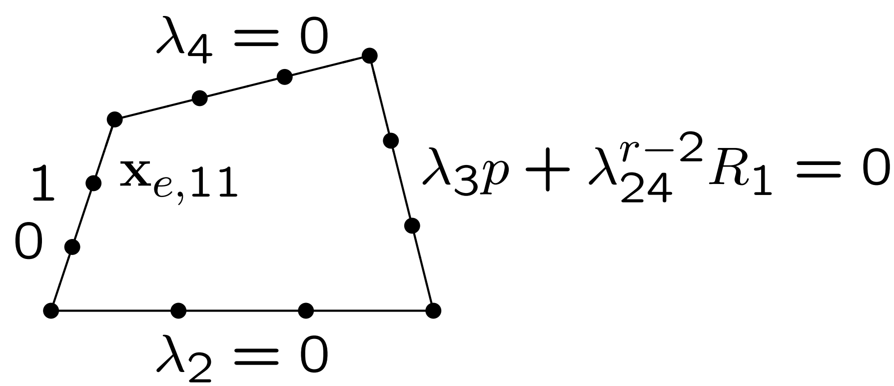
Shape functions. For example, we construct $\varphi_{e,11}(\mathbf{x})$, which is 1 at $\mathbf{x}_{e,11}$ and vanishes at all other nodal points

For some $p \in \mathbb{P}_{r-3}(E)$ (take $p = 0$ if $r = 2$), let

$$\phi_{e,11} = \lambda_2\lambda_4\Big(\lambda_3 p + \lambda_{24}^{r-2}R_1\Big) \in \mathcal{DS}_r(E),$$

with p satisfying conditions

$$p(\mathbf{x}_{e,1i}) = -\frac{\lambda_{24}^{r-2}(\mathbf{x}_{e,1i})}{\lambda_3(\mathbf{x}_{e,1i})}, \quad \forall i = 2, \ldots, r-1$$

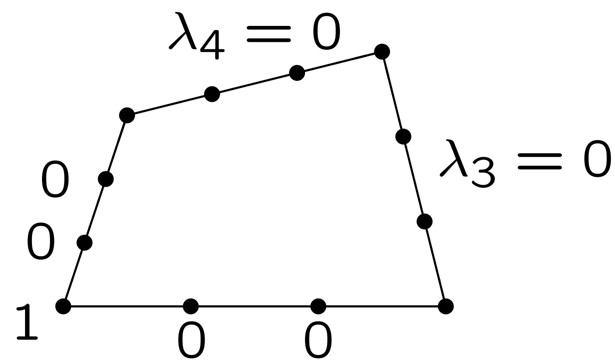


Nodal basis functions. We can prove that $\phi_{e,11}(\mathbf{x}_{e,11}) \neq 0$, and define $\varphi_{e,11}$ by normalization

Remark: Normalization means to remove the interior DoFs (subtract multiples of $\varphi_{E,i}$), and then scale the nodal value to 1

Shape functions. For example, we define the shape function for $\mathbf{x}_{v,1}$

$$\phi_{v,1}(\mathbf{x}) = \lambda_3(\mathbf{x})\,\lambda_4(\mathbf{x}) - \sum_{k=1}^{2}\sum_{\ell=1}^{r-1} \lambda_3(\mathbf{x}_{e,k\ell})\,\lambda_4(\mathbf{x}_{e,k\ell})\,\varphi_{e,k\ell}(\mathbf{x})$$



Our shape functions vanish at all of the edge nodes, and $\phi_{v,i}(\mathbf{x}_{v,j}) = 0$ if $i \neq j$ and is positive otherwise

Nodal basis functions.

$$\varphi_{v,i}(\mathbf{x}) = \frac{\phi_{v,i}(\mathbf{x}) - \sum_{k=1}^{\dim \mathbb{P}_{r-4}(E)} \phi_{v,i}(\mathbf{x}_{E,k})\,\varphi_{E,k}(\mathbf{x})}{\phi_{v,i}(\mathbf{x}_{v,i})}, \qquad i = 1, 2, 3, 4$$

Theorem. Assume that

- $v \in W_2^l(\Omega)$
- $\mathcal{T}_h$ is uniformly shape regular with parameter σ_*
- For every $E \in \mathcal{T}_h$, the zero set $\mathcal{L}_{24}$ intersects e_1 and e_3, and $\mathcal{L}_{13}$ intersects e_2 and e_4
- R_{13} and R_{24} are uniformly differentiable functions of the vertices of E up to order m

Then there exists a constant $C = C(r, \sigma_*) > 0$ such that for all functions $v \in W_p^l(E)$,

$$||v - \mathcal{I}_h^r v||_{W_2^m(E)} \leq C\, h_E^{l-m} |v|_{W_2^l(E)}, \quad 0 \leq m \leq l \leq r+1$$

Moreover, there exists a constant $C = C(r, \sigma_*) > 0$, independent of $h = \max_{E \in \mathcal{T}_h} h_E$, such that for all functions $v \in W_2^l(\Omega)$,

$$\Big(\sum_{E \in \mathcal{T}_h} ||v - \mathcal{I}_h^r v||^2_{W_2^m(E)} \Big)^{1/2} \leq C\, h^{l-m}\, |v|_{W_2^l(\Omega)}, \quad 0 \leq m \leq l \leq r+1$$

Numerical Results

Test problem. We consider the following problem defined on the unit square $\Omega = [0,1]^2$ with the coefficient $\mathbf{a}$ being the 2×2 identity matrix

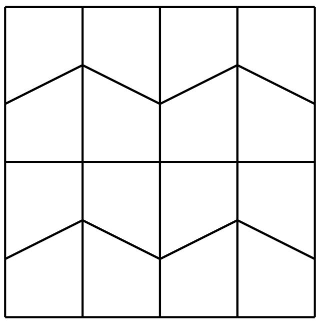
$$\begin{aligned} -\nabla \cdot (\mathbf{a} \nabla p) &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

Weak form. Find $p \in H_0^1(\Omega)$ such that

$$(\mathbf{a}\nabla p, \nabla q) = (f, q), \quad \forall q \in H_0^1(\Omega),$$

Exact solution. $u(x,y) = \sin(\pi x)\sin(\pi y)$

Mesh. A mesh of n^2 trapezoids of base h and one pair of parallel edges of size $0.75h$ and $1.25h$. Finer meshes are constructed by repeating the same pattern over the domain



$\mathcal{T}_h$

L^2-errors and convergence rates for $\mathbb{P}_{r,r}$, $\mathcal{DS}_r$, and $\mathcal{S}_r$ spaces on trapezoidal meshes

	$r = 2$		$r = 3$		$r = 4$		$r = 5$	
n	error	rate	error	rate	error	rate	error	rate
				$\mathbb{P}_{r,r}$ on $\mathcal{T}_h$ meshes				
8	3.329e-04	2.99	9.740e-06	3.99	2.382e-07	4.99	5.076e-09	5.99
12	9.888e-05	2.99	1.928e-06	3.99	3.142e-08	5.00	4.462e-10	6.00
16	4.176e-05	3.00	6.107e-07	4.00	7.459e-09	5.00	7.946e-11	6.00
24	1.238e-05	3.00	1.207e-07	4.00	9.827e-10	5.00	6.979e-12	6.00
				$\mathcal{S}_r$ on $\mathcal{T}_h$ meshes				
8	5.714e-04	2.92	4.844e-04	2.89	2.612e-05	3.72	2.005e-06	4.13
12	1.731e-04	2.94	1.482e-04	2.92	6.084e-06	3.59	3.884e-07	4.05
16	7.409e-05	2.95	6.383e-05	2.93	2.265e-06	3.43	1.234e-07	3.99
24	2.254e-05	2.94	1.963e-05	2.91	5.984e-07	3.28	2.516e-08	3.92
32	9.799e-06	2.90	8.635e-06	2.85	2.408e-07	3.16	8.342e-09	3.84
64	1.440e-06	2.70	1.332e-06	2.61	2.862e-08	3.05	6.644e-10	3.56
				$\mathcal{DS}_r$ on $\mathcal{T}_h$ meshes				
8	3.492e-04	3.00	3.897e-05	4.07	2.187e-06	5.00	8.896e-08	5.96
12	1.036e-04	3.00	7.457e-06	4.08	2.889e-07	4.99	7.870e-09	5.98
16	4.373e-05	3.00	2.313e-06	4.07	6.868e-08	4.99	1.404e-09	5.99
24	1.296e-05	3.00	4.469e-07	4.05	9.058e-09	5.00	1.235e-10	6.00

H^1-seminorm errors and convergence rates for $\mathbb{P}_{r,r}$, $\mathcal{DS}_r$, and $\mathcal{S}_r$ spaces on trapezoidal meshes

	$r=2$		$r=3$		$r=4$		$r=5$	
n	error	rate	error	rate	error	rate	error	rate
$\mathbb{P}_{r,r}$ on $\mathcal{T}_h$ meshes								
8	1.734e-02	2.00	7.206e-04	2.99	2.310e-05	3.99	6.083e-07	4.99
12	7.710e-03	2.00	2.139e-04	3.00	4.570e-06	4.00	8.021e-08	5.00
16	4.337e-03	2.00	9.027e-05	3.00	1.447e-06	4.00	1.904e-08	5.00
24	1.928e-03	2.00	2.676e-05	3.00	2.859e-07	4.00	2.509e-09	5.00
$\mathcal{S}_r$ on $\mathcal{T}_h$ meshes								
8	2.413e-02	1.94	1.834e-02	1.90	1.818e-03	2.65	1.537e-04	3.18
12	1.105e-02	1.93	8.572e-03	1.88	6.582e-04	2.51	4.483e-05	3.04
16	6.432e-03	1.88	5.091e-03	1.81	3.345e-04	2.35	1.945e-05	2.90
24	3.104e-03	1.80	2.560e-03	1.70	1.360e-04	2.22	6.370e-06	2.75
32	1.920e-03	1.67	1.643e-03	1.54	7.378e-05	2.12	3.029e-06	2.58
64	7.097e-04	1.34	6.602e-04	1.23	1.776e-05	2.03	5.953e-07	2.26
$\mathcal{DS}_r$ on $\mathcal{T}_h$ meshes								
8	1.836e-02	2.01	2.517e-03	3.02	1.625e-04	3.99	7.384e-06	4.99
12	8.143e-03	2.00	7.400e-04	3.02	3.216e-05	4.00	9.757e-07	4.99
16	4.577e-03	2.00	3.109e-04	3.01	1.018e-05	4.00	2.318e-07	5.00
24	2.033e-03	2.00	9.170e-05	3.01	2.012e-06	4.00	3.056e-08	5.00

2. Direct Mixed Spaces

$$\mathbf{V}_r(E) = \mathbb{P}_r(E)^2 \oplus \mathbb{S}_r^{\mathbf{V}}(E)$$

Polynomials plus supplements

We get both full and reduced $H(\text{div})$-approximation spaces from any direct serendipity family by de Rham:

$$\mathbb{R} \hookrightarrow \mathcal{DS}_{r+1}(E) \xrightarrow{\text{curl}} \mathbf{V}_r^{\text{full}}(E) \xrightarrow{\text{div}} \mathbb{P}_r(E) \longrightarrow 0$$

$$\mathbb{R} \hookrightarrow \mathcal{DS}_{r+1}(E) \xrightarrow{\text{curl}} \mathbf{V}_r^{\text{red}}(E) \xrightarrow{\text{div}} \mathbb{P}_{r-1}(E) \longrightarrow 0$$

The image of one map is the kernel of the next. DoFs mapped properly

Image of div.

$$\operatorname{div} \mathbf{x}\mathbb{P}_r = \mathbb{P}_r \quad \text{and} \quad \operatorname{div} \mathbf{x}\mathbb{P}_{r-1} = \mathbb{P}_{r-1}$$

Spaces. For $\mathbf{V}_r^{\text{full}}(E)$,

$$\begin{aligned}
\mathbf{V}_r^{\text{full}}(E) &= \operatorname{curl} \mathcal{DS}_{r+1}(E) \oplus \mathbf{x}\mathbb{P}_r \\
&= \operatorname{curl} \mathbb{P}_{r+1}(E) \oplus \operatorname{curl} \mathbb{S}_{r+1}^{\mathcal{DS}}(E) \oplus \mathbf{x}\mathbb{P}_r \\
&= \operatorname{curl} \mathbb{P}_{r+1}(E) \oplus \mathbf{x}\mathbb{P}_r \oplus \operatorname{curl} \mathbb{S}_{r+1}^{\mathcal{DS}}(E) \\
&= \operatorname{curl} \mathbb{P}_{r+1}(E) \oplus \mathbf{x}\mathbb{P}_{r-1} \oplus \mathbf{x}\tilde{\mathbb{P}}_r \oplus \operatorname{curl} \mathbb{S}_{r+1}^{\mathcal{DS}}(E) \\
&= \mathbb{P}_r^2(E) \oplus \mathbf{x}\tilde{\mathbb{P}}_r \oplus \mathbb{S}_r^{\mathbf{V}}(E)
\end{aligned}$$

where we identify

$$\mathbb{S}_r^{\mathbf{V}}(E) = \operatorname{curl} \mathbb{S}_{r+1}^{\mathcal{DS}}(E)$$

The reduced space is similar

New spaces. All the direct serendipity elements give new direct mixed finite elements. These can be constructed without requiring any mappings from a reference element

AC spaces. We recover the AC spaces (Arbogast & Correa 2016) by a special choice of supplements $\mathbb{S}_r^{\mathcal{DS}}(E)$ that are mapped

$$\text{curl(mapped } \phi) = \text{Piola-mapped(curl}\,\phi)$$

DoFs. We define $\pi_E \boldsymbol{\psi}$ in terms of the DoFs for $\boldsymbol{\psi} \in \mathbf{V}_r^s(E)$, $s = r-1, r$,

$$\begin{aligned}
&\int_{e_i} \boldsymbol{\psi} \cdot \nu_i\, p\, d\sigma, && \forall p \in \mathbb{P}_r(e_i),\ i = 1,2,3,4,\\
&\int_E \boldsymbol{\psi} \cdot \nabla q\, dx, && \forall q \in \mathbb{P}_s(E),\\
&\int_E \boldsymbol{\psi} \cdot \mathbf{v}\, dx, && \forall p \in \mathbb{B}_r^{\mathbf{V}}(E),
\end{aligned}$$

where

$$\mathbb{B}_{r+1}(E) = \lambda_1\lambda_2\lambda_3\lambda_4\mathbb{P}_{r-3}(E) \quad \text{and} \quad \mathbb{B}_r^{\mathbf{V}}(E) = \text{curl}\,\mathbb{B}_{r+1}(E)$$

1. π satisfies the commuting diagram property:

$$\mathcal{P}_{W_s}\nabla\cdot\mathbf{v} = \nabla\cdot\pi\mathbf{v}$$

2. Since π_E is bounded in, say, H^1, suppose that $\mathcal{T}_h$ is uniformly shape regular as $h \to 0$. Then there is a constant $C > 0$, independent of h, such that

$$\begin{aligned} \|\mathbf{v}-\pi\mathbf{v}\|_{0,\Omega} &\le C\,\|\mathbf{v}\|_{k,\Omega}\,h^k, && k = 1,\dots,r+1 \\ \|\nabla\cdot(\mathbf{v}-\pi\mathbf{v})\|_{0,\Omega} &\le C\,\|\nabla\cdot\mathbf{v}\|_{k,\Omega}\,h^k, && k = 0,1,\dots,s+1 \\ \|p-\mathcal{P}_{W_s}p\|_{0,\Omega} &\le C\,\|p\|_{k,\Omega}\,h^k, && k = 0,1,\dots,s+1 \end{aligned}$$

3. The discrete inf-sup condition

$$\sup_{\mathbf{v}_h\in\mathbf{V}_r}\frac{(w_h,\nabla\cdot\mathbf{v}_h)}{\|\mathbf{v}_h\|_{H(\mathrm{div})}} \ge \gamma\,\|w_h\|_{0,\Omega}, \qquad \forall w_h\in W_s$$

holds for some $\gamma > 0$ independent of h

Test problem. We consider the following problem defined on the unit square $\Omega = [0,1]^2$ with the coefficient $\mathbf{a}$ being the 2×2 identity matrix

$$\begin{aligned} -\nabla \cdot (\mathbf{a}\nabla p) &= f \quad \text{in } \Omega \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

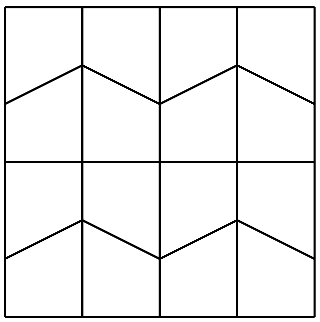
Weak form. Setting

$$\mathbf{u} = -\mathbf{a}\nabla p,$$

find $\mathbf{u} \in H(\text{div};\Omega)$ and $p \in L^2(\Omega)$ such that

$$\begin{aligned} (\mathbf{a}^{-1}\mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) &= 0, && \forall \mathbf{v} \in H(\text{div};\Omega) \\ (\nabla \cdot \mathbf{u}, w) &= (f, w), && \forall w \in L^2(\Omega) \end{aligned}$$

Mesh.



$\mathcal{T}_h$

Errors and convergence rates for fully direct mixed spaces on $\mathcal{T}_h$.

n	$\|\|p - p_h\|\|$ error	rate	$\|\mathbf{u} - \mathbf{u}_h\|$ error	rate	$\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|$ error	rate
$r = 1$, reduced H(div)-approximation						
4	1.670e-01	—	2.609e-01	—	3.163e-00	—
8	8.271e-02	1.01	6.803e-02	1.96	1.612e-00	0.98
16	4.117e-02	1.00	1.719e-02	1.99	8.099e-01	1.00
32	2.056e-02	1.00	4.309e-03	2.00	4.054e-01	1.00
$r = 2$, reduced H(div)-approximation						
4	3.079e-02	—	2.319e-02	—	6.067e-01	—
8	7.847e-03	1.98	2.906e-03	3.00	1.549e-01	1.98
16	1.972e-03	2.00	3.633e-04	3.00	3.892e-02	2.00
32	4.936e-04	2.00	4.543e-05	3.00	9.742e-03	2.00
$r = 1$, full H(div)-approximation						
4	3.079e-02	—	5.562e-02	—	6.067e-01	—
8	7.847e-03	1.98	1.350e-02	2.02	1.549e-01	1.98
16	1.972e-03	2.00	3.355e-03	2.01	3.892e-02	2.00
32	4.936e-04	2.00	8.378e-04	2.00	9.742e-03	2.00
$r = 2$, full H(div)-approximation						
4	4.081e-03	—	7.198e-03	—	8.050e-02	—
8	5.201e-04	2.98	9.105e-04	2.99	1.026e-02	2.98
16	6.533e-05	3.00	1.141e-04	3.00	1.289e-03	3.00
32	8.176e-06	3.00	1.428e-05	3.00	1.614e-04	3.00

3. Application to Porous Media with quadrilateral meshes

Direct Mixed Methods and Serendipity Spaces in Enriched Galerkin

General IMPES/IMPEC Formulation

(Lee & Wheeler 2017)

Mass must be conserved

Implicit Pressure for Flow. Solve for the pressure and velocity using mixed finite element methods

- Use AC_1^{full} (or AC_2^{full}) spaces on quadrilaterals (Arbogast & Correa 2016)

Explicit Saturation/Concentration for Transport. Solve for the concentration or saturation using enriched Galerkin (Sun & Liu 2009)

- Use direct serendipity spaces $\mathcal{DS}_2$ on quadrilaterals enriched with piecewise discontinuous constants
- Discontinuous Galerkin NIPG bilinear form
- Need entropy stabilization (Guermond, Pasquetti & Popov 2011)

Tracer Transport

Tracer Transport — 1

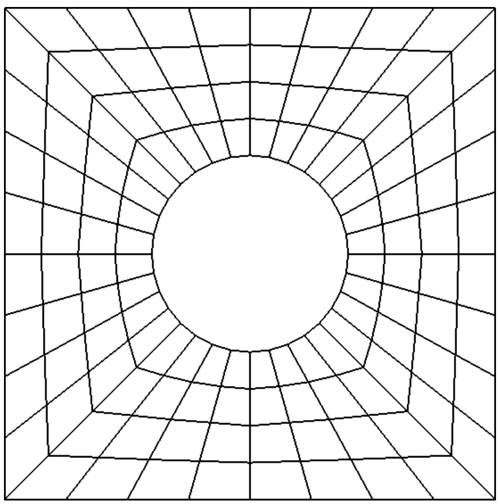
Governing equations.

Flow: $\begin{cases} \mathbf{u} = -\mathbf{K}\nabla p \\ \nabla\cdot\mathbf{u} = q \end{cases}$ Direct Mixed FE

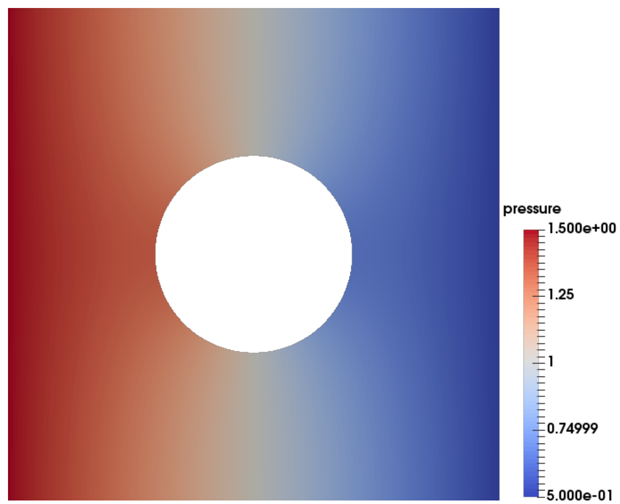
Transport: $\phi\dfrac{\partial c}{\partial t} + \nabla\cdot(c\mathbf{u} - \mu_h \nabla c) = q_c$ EG with Entropy Stability

We take $\phi = 1$, set $k = 1$, and $q = q_c = 0$

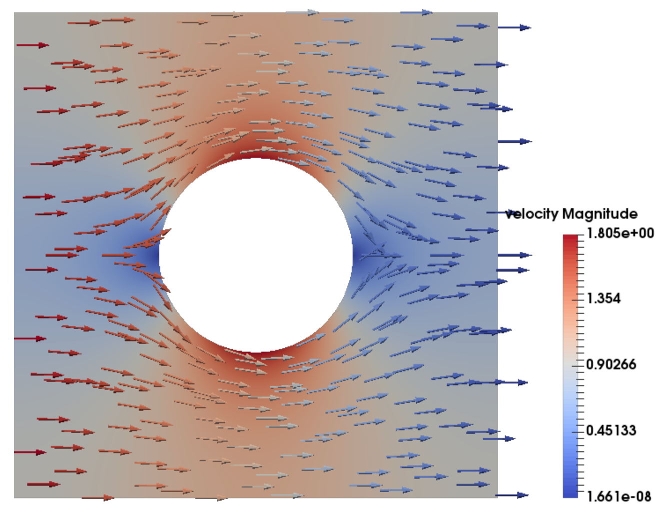
Flow solution.



Coarse mesh
128 elements



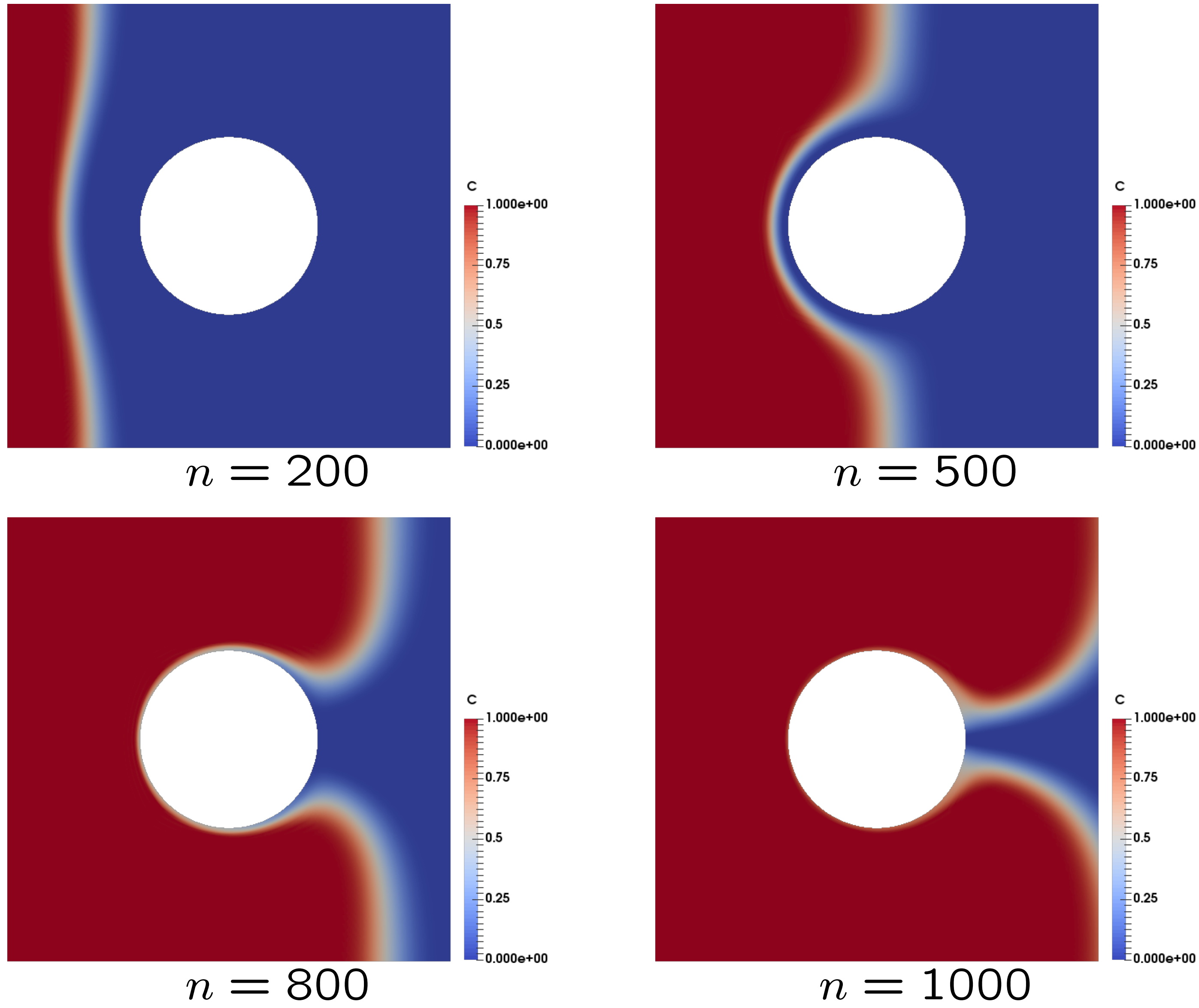
Pressure



Velocity

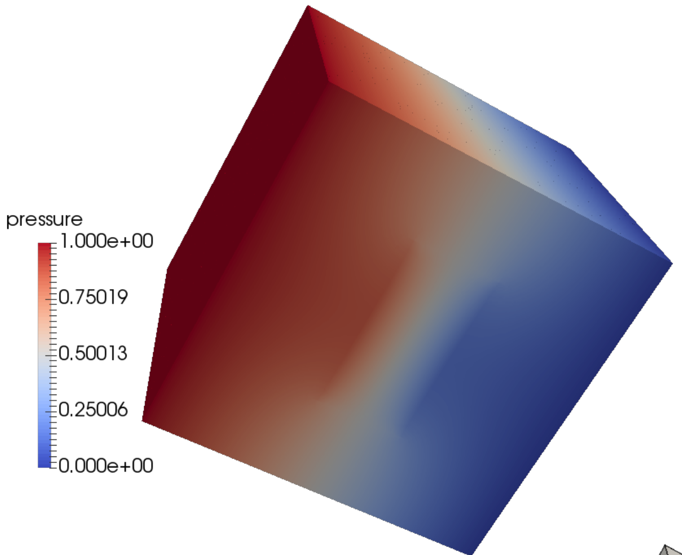
8192 elements

Concentration. (8192 elements, $\Delta t = 0.1h_{\text{min}}$)

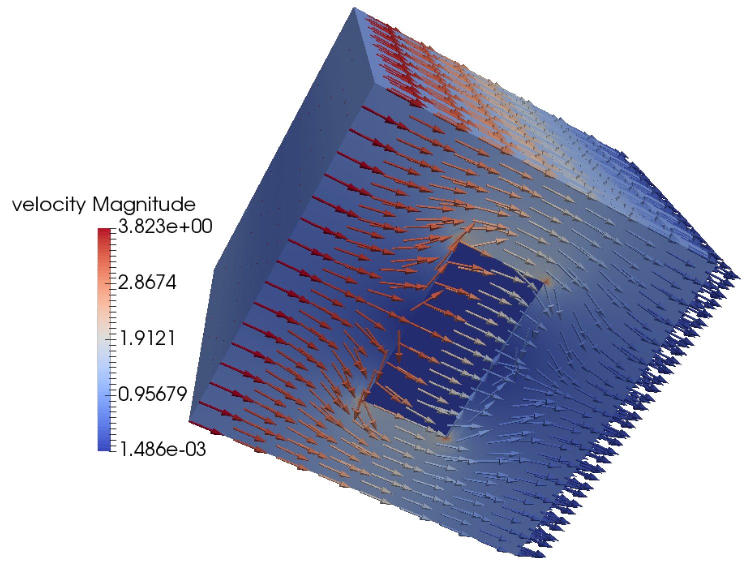


Similar to Sun and Liu 2009 (but we use many fewer DoFs) and Lee, Lee & Wheeler 2016 (but follow a round hole)

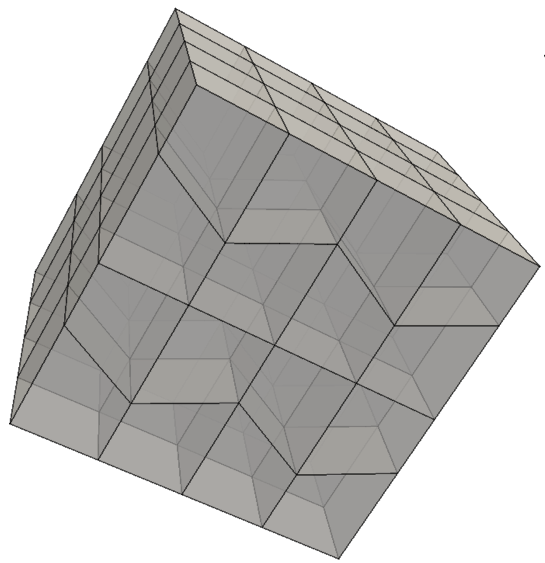
Flow solution. on $32 \times 32 \times 32$ mesh of truncated pillars.



Pressure

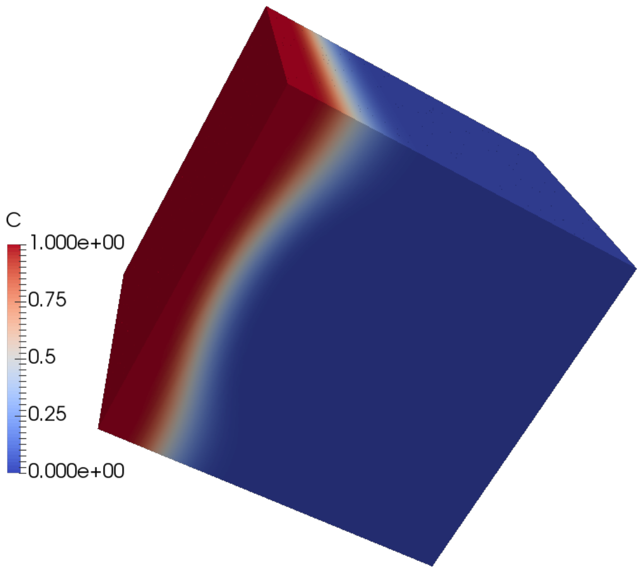


Velocity

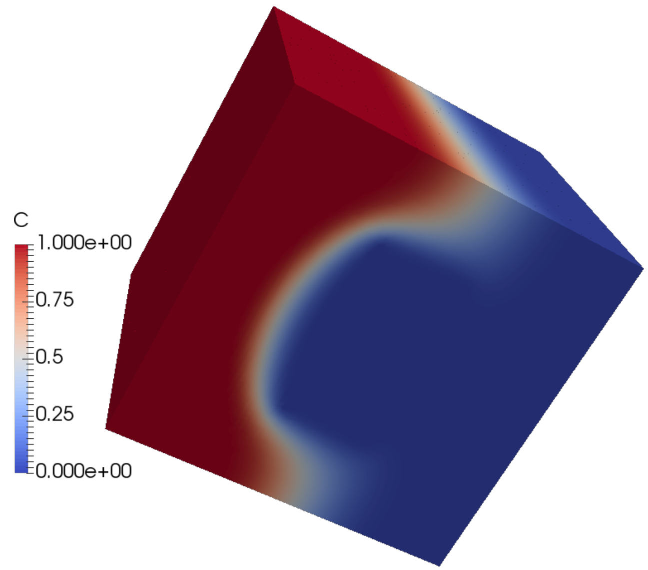


Coarse mesh, $4 \times 4 \times 4$

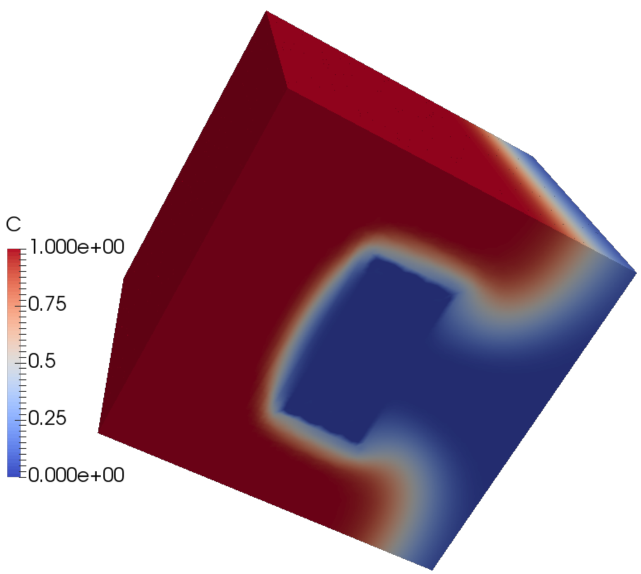
Concentration. ($32 \times 32 \times 32$ mesh, $\Delta t = 0.1 h_{\text{min}}$)



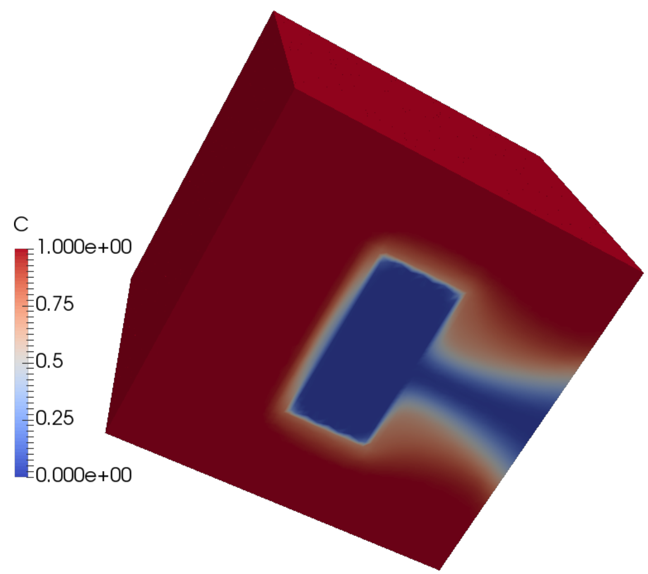
$n = 40$



$n = 100$



$n = 160$



$n = 250$

Two-Phase Flow

(Hoteit & Firoozabadi 2008)

Variables.

- Potentials $\Phi_\alpha = p_\alpha - \rho_\alpha g z$, $\alpha = \mathrm{w}, \mathrm{n}$ (wetting and nonwetting)
- Capillary potential $\Phi_\mathrm{c} = \Phi_\mathrm{n} - \Phi_\mathrm{w} = p_\mathrm{c}(S_\mathrm{w}) + (\rho_\mathrm{n} - \rho_\mathrm{w}) g z$
- Total velocity $\mathbf{u}_\mathrm{t} = \mathbf{u}_\mathrm{w} + \mathbf{u}_\mathrm{n} = \mathbf{u}_\mathrm{a} + \mathbf{u}_\mathrm{c}$ (advective and capillary)

Flow.

$$\begin{cases} \mathbf{u}_\mathrm{a} = -\lambda_\mathrm{t} \mathbf{K} \nabla \Phi_\mathrm{w} \\ \nabla \cdot \mathbf{u}_\mathrm{a} = q_\mathrm{t} - \nabla \cdot \mathbf{u}_\mathrm{c} \end{cases}$$

Transport.

$$\phi \frac{\partial S_\mathrm{w}}{\partial t} = q_\mathrm{w} - \nabla \cdot (f_\mathrm{w} \mathbf{u}_\mathrm{a})$$

Capillary Flux—A New Approach. (Arbogast & Tao 2016)

$$\mathbf{u}_\mathrm{c} = -\lambda_\mathrm{n}(S_\mathrm{w}) \mathbf{K} \nabla \Phi_\mathrm{c}$$

Let $\hat{\zeta} = \lambda_\mathrm{n} \Phi_\mathrm{c}$, and solve $\mathbf{u}_\mathrm{c} \in \mathbf{V}_h$ and $\hat{\zeta} \in M_h$ such that

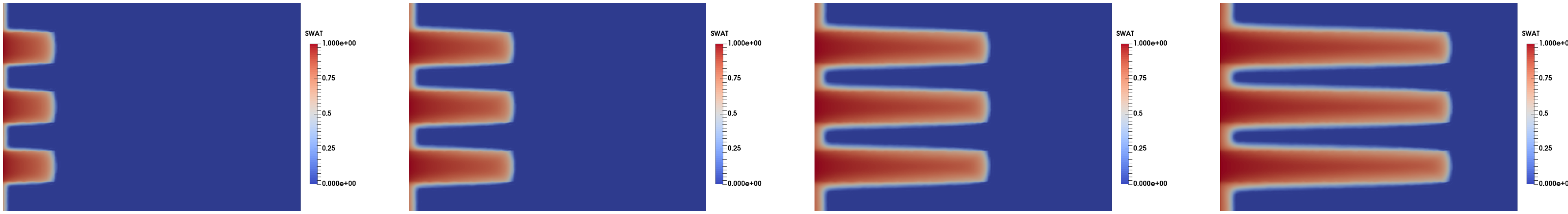
$$\int_E \mathbf{K}^{-1} \mathbf{u}_\mathrm{c} \cdot \mathbf{v} = \int_E \Phi_\mathrm{c}(S_\mathrm{w}) \nabla \cdot (\lambda_\mathrm{n}(S_\mathrm{w}) \mathbf{v}) - \int_{\partial E} \hat{\zeta} \mathbf{v} \cdot \nu \qquad \forall \mathbf{v} \in \mathbf{V}_h$$

$$\sum_E \int_{\partial E} \mathbf{u}_\mathrm{c} \cdot \nu \, \mu = 0 \qquad \forall \mu \in M_h$$

Cleanly handles the degeneracy $\lambda_\mathrm{n}(S_\mathrm{w}) = 0$

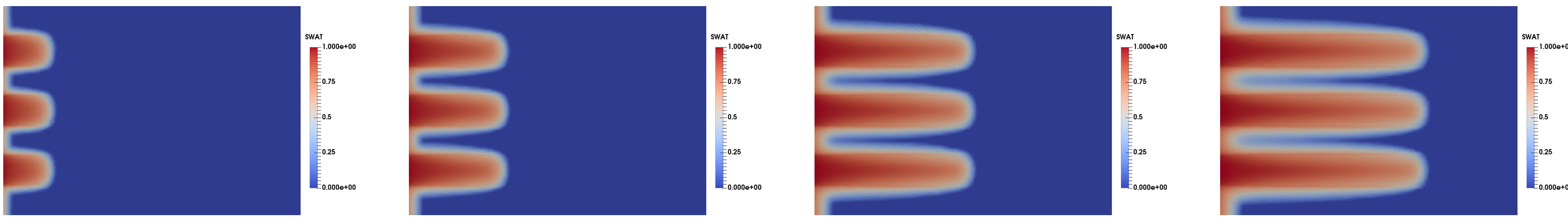
Two-Phase Flow — 1

Without Capillary Pressure.



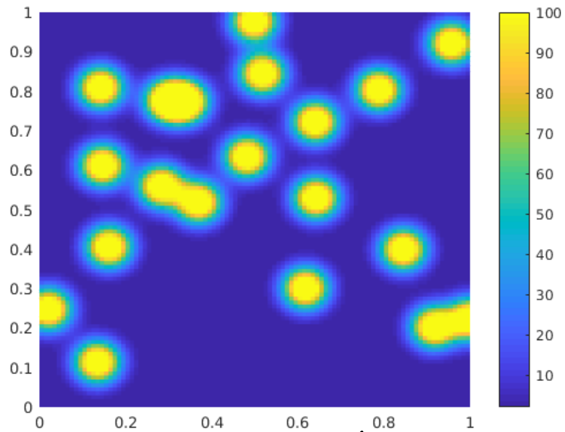
$n = 50$ $n = 125$ $n = 250$ $n = 375$

With Capillary Pressure.

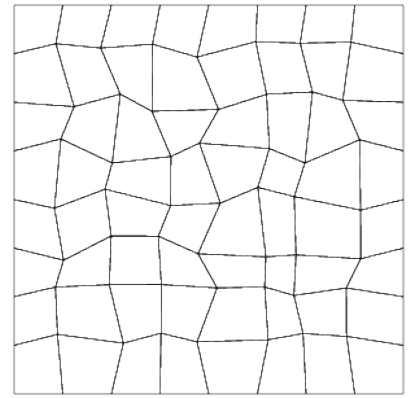


$n = 50$ $n = 125$ $n = 250$ $n = 375$

Heterogeneous permeability.

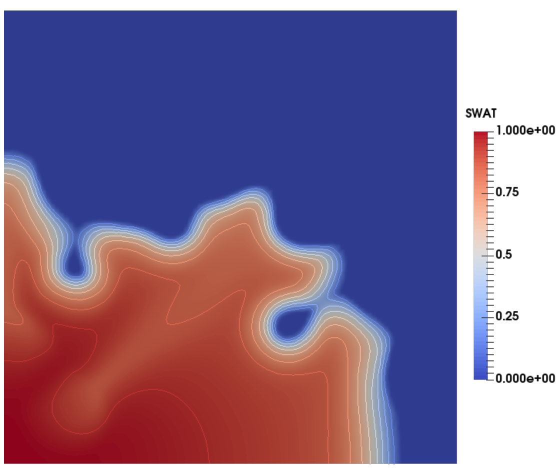


permeability $k^{-1} \in [2, 100]$

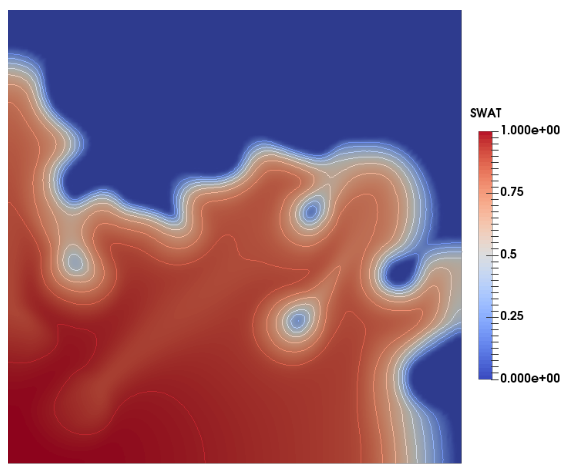


8×8 random quadrilateral mesh

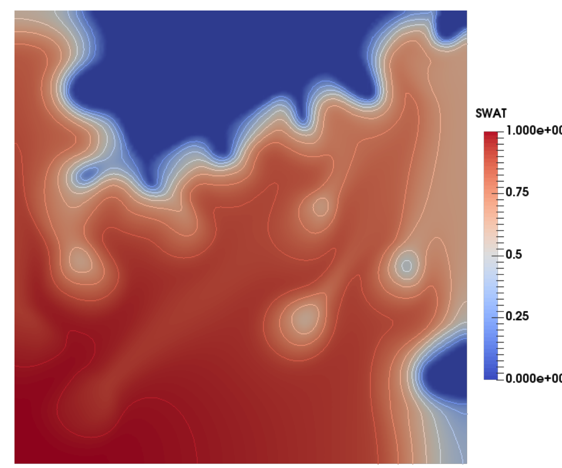
Saturation.



$n = 1000$



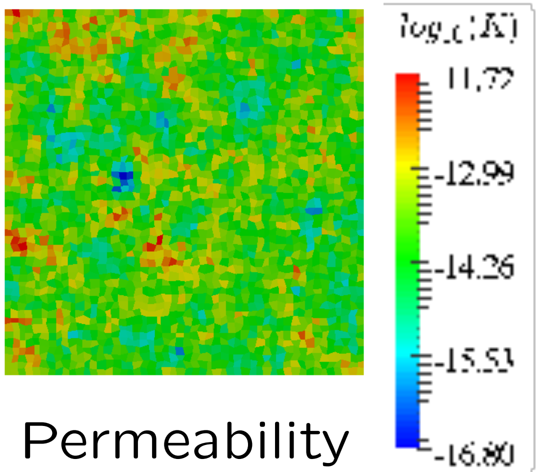
$n = 1500$



$n = 2000$

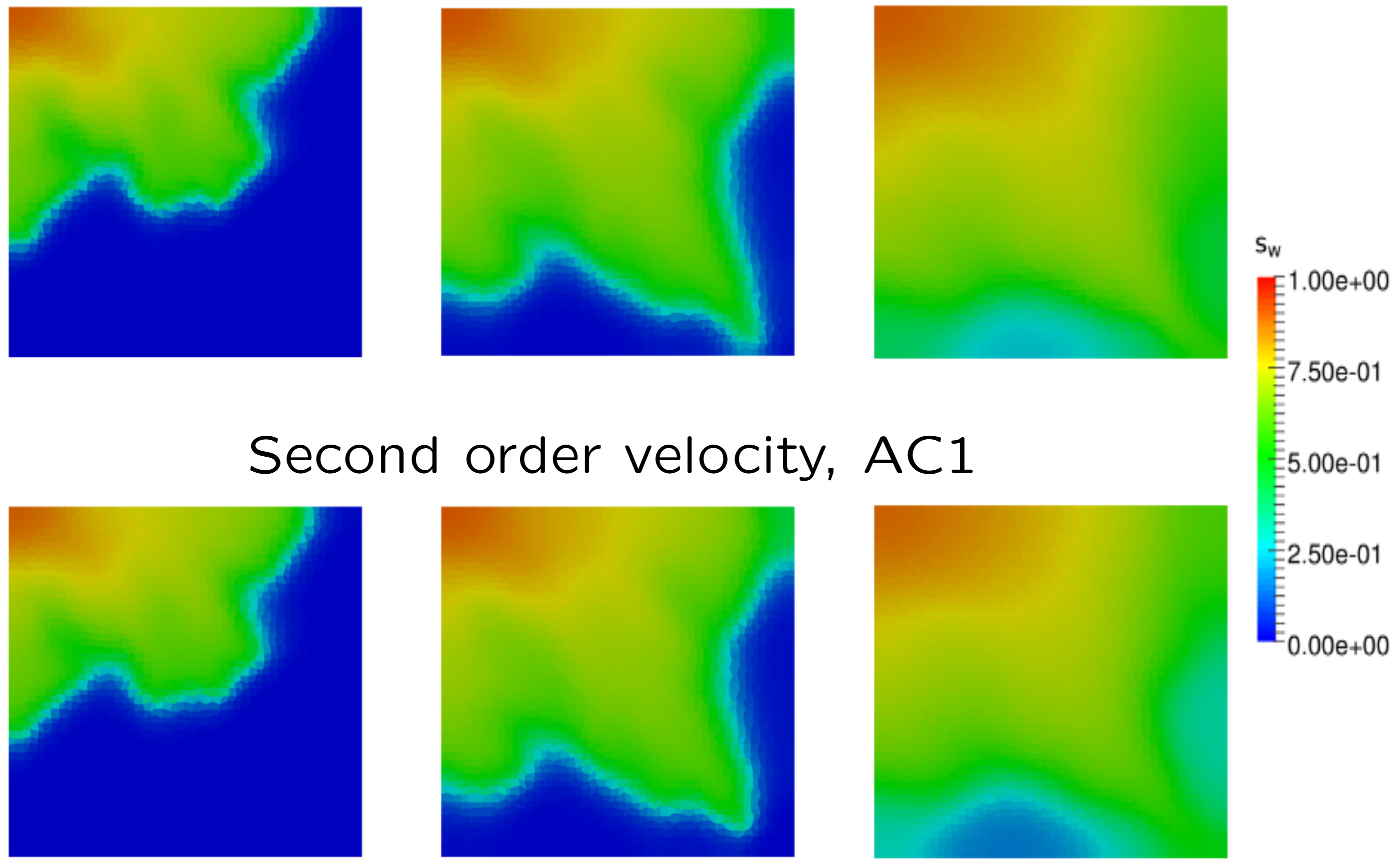
100×100 random mesh of quadrilaterals

Quarter 5 spot pattern



Permeability on Quadrilateral Mesh

First order velocity, AC0



$t = 200$ days $t = 350$ days $t = 600$ days

Flow uses global pressure; transport solved with WENO

> The higher order velocity makes a difference

4. Summary and Conclusions

Summary and Conclusions

1. Many families of direct serendipity spaces found for quadrilaterals
   - Constructed explicit basis
2. New direct mixed finite element spaces found for quadrilaterals
   - Arise from the de Rham theory
   - We found the direct serendipity space giving AC spaces
3. Good results for porous media applications on quadrilateral meshes
   - Flow: Direct mixed methods
   - Transport: Enriched Galerkin with direct serendipity elements
   - Tracer transport
   - Two-phase flow (new capillary flux)

References

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2. T. Arbogast and Zhen Tao: A direct mixed-enriched Galerkin method on quadrilaterals for two-phase Darcy flow. Computational Geosci., 23(5):1141–1160, 2019.
3. T. Arbogast and Zhen Tao: Construction of H(div)-conforming mixed finite elements on cuboidal hexahedra. Numer. Math., 142(1):1–32, 2019.
4. T. Arbogast and M. R. Correa: Two families of H(div) mixed finite elements on quadrilaterals of minimal dimension. SIAM J. Numer. Anal., 54(6):3332–3356, 2016.
5. T. Arbogast, Ch.-S. Huang, and Xikai Zhao. Von Neumann stable, implicit, high order, finite volume WENO schemes. In SPE Reservoir Simulation Conference 2019, pages 1-16, Galveston, Texas, April 2019. Society of Petroleum Engineers. SPE-193817-MS.