Direct Serendipity and Mixed Finite Elements on Convex Polygons

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Motivation

Flexible meshing using polygons is useful in many contexts, e.g.,

- topology optimization;
- crack fracture and propagation;
- elasticity;
- composite materials;
- computational fluid dynamics.

Conforming finite elements are useful for solving PDEs.

Moreover, one might use mimetic or DG methods to solve the PDE but use conforming interpolation to better:

- approximate the nonlinearities in a PDE;
- transfer the solution between equations in a system of PDEs;
- visualize the solution.

They also provide a framework for approximation of functions:

- interpolation of data;
- visualization.

With *known* error bounds.

Problem. Not many conforming finite elements on polygons are known.

Extend the theory of serendipity and mixed finite elements on rectangles to polygons

Serendipity finite elements. $\mathcal{S}_r$ on rectangle $\hat{E}_4$:

- H^1-conforming
- Approximate to $\mathcal{O}(h^{r+1})$ with minimal # degrees of freedom (DoFs)

BDM mixed finite elements. BDM_r on rectangle $\hat{E}_4$:

- $H(\mathrm{div})$-conforming
- Approximate velocity to $\mathcal{O}(h^{r+1})$ with minimal # of DoFs

Related by de Rham complex. (Arnold, Falk & Winther 2006)

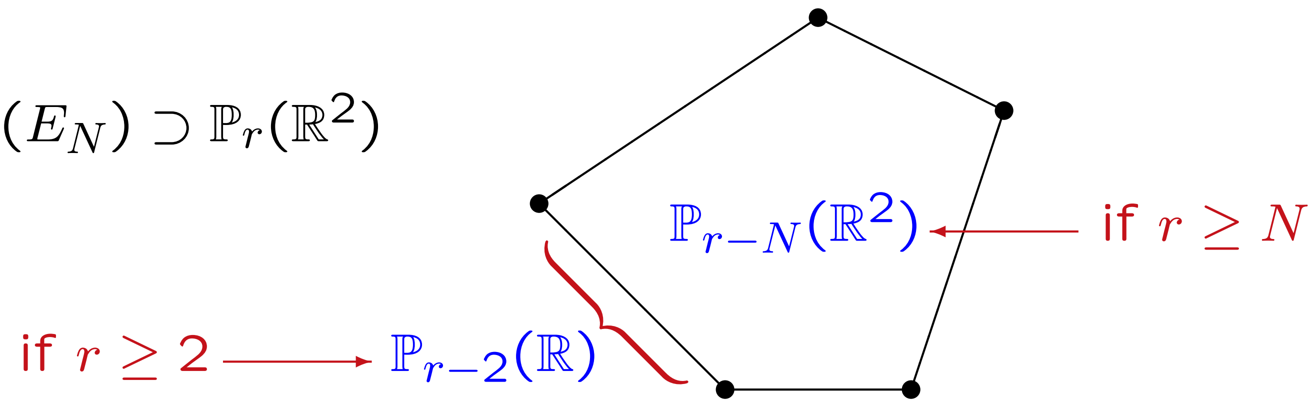
$$\mathbb{R} \hookrightarrow \mathcal{S}_{r+1}(\hat{E}_4) \xrightarrow{\mathrm{curl}} \mathrm{BDM}_r(\hat{E}_4) \xrightarrow{\mathrm{div}} \mathbb{P}_{r-1}(\hat{E}_4) \longrightarrow 0$$

Problem. Lose accuracy when mapped to a quadrilateral E_4

Goals. Define direct finite element spaces that

- Include polynomials $\mathbb{P}_r$ *directly* in the space (for approximation)
- Use minimal number of DoFs
- Apply to convex polygons E_N with N sides

Require $\mathcal{DS}_r(E_N) \supset \mathbb{P}_r(\mathbb{R}^2)$



DoFs required for H^1-Conformity ($N \geq 3$, $r \geq 1$)

Object	Object Count	DoFs per Object	Total DoFs
vertex	N	1	N
interior edge	N	$\dim \mathbb{P}_{r-2}(\mathbb{R})$	$N(r-1)$
interior cell	1	$\dim \mathbb{P}_{r-N}(\mathbb{R}^2)$	$\underbrace{\frac{1}{2}(r-N+2)(r-N+1)}_{\text{provided } r \geq N-2}$

Goal: Define the Supplemental Space $\mathbb{S}_r^{\mathcal{DS}}(E_N)$

The DoF counts imply

$$\mathcal{DS}_r(E_N) = \mathbb{P}_r(E_N) \oplus \mathbb{S}_r^{\mathcal{DS}}(E_N)$$

where

$$\mathbb{P}_r(E_N) \cap \mathbb{S}_r^{\mathcal{DS}}(E_N) = \emptyset$$

$$\dim \mathbb{S}_r^{\mathcal{DS}}(E_N) = \begin{cases} \frac{1}{2}N(N-3), & r \geq N-2 \\ Nr - \frac{1}{2}(r+2)(r+1) < \frac{1}{2}N(N-3), & r < N-2 \end{cases}$$

Counterintuitive observation. The case $r \geq N-2$ is easier!

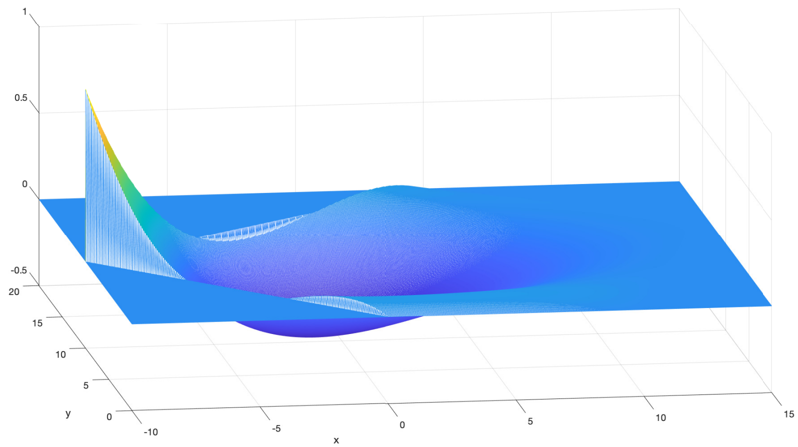
Cannot build higher order spaces from $r = 1$ (barycentric coordinates).

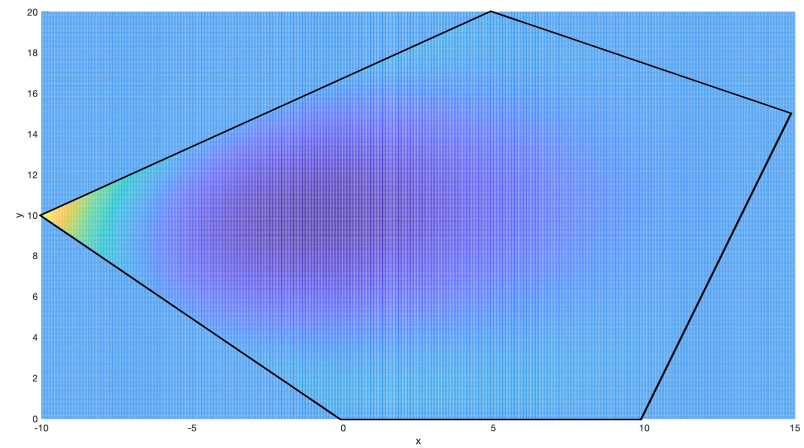
- For $r < N-2$: We will define $\mathcal{DS}_r(E_N) \subset \mathcal{DS}_{N-2}(E_N)$.
- For $r \geq N-2$: For each of the N edges, there are $N-3$ nonadjacent edges. That is, the number of nonadjacent edge pairs is

$$\frac{1}{2}N(N-3) = \dim \mathbb{S}_r^{\mathcal{DS}}(E_N)$$

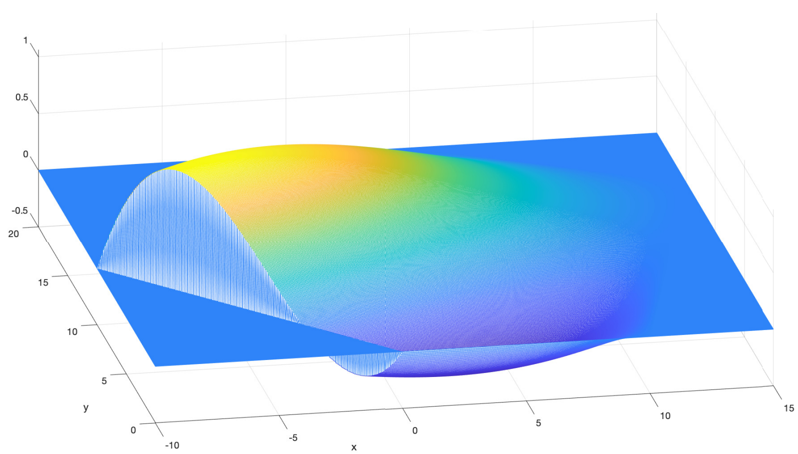
  Supplemental basis functions are associated to pairs of edges!

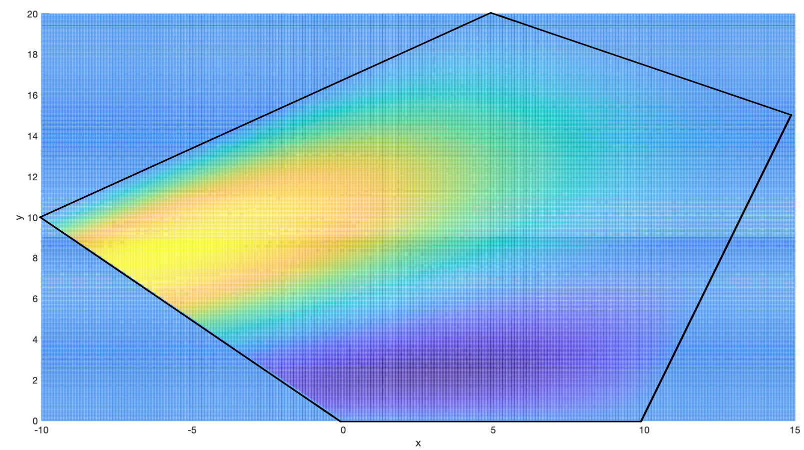
Vertex





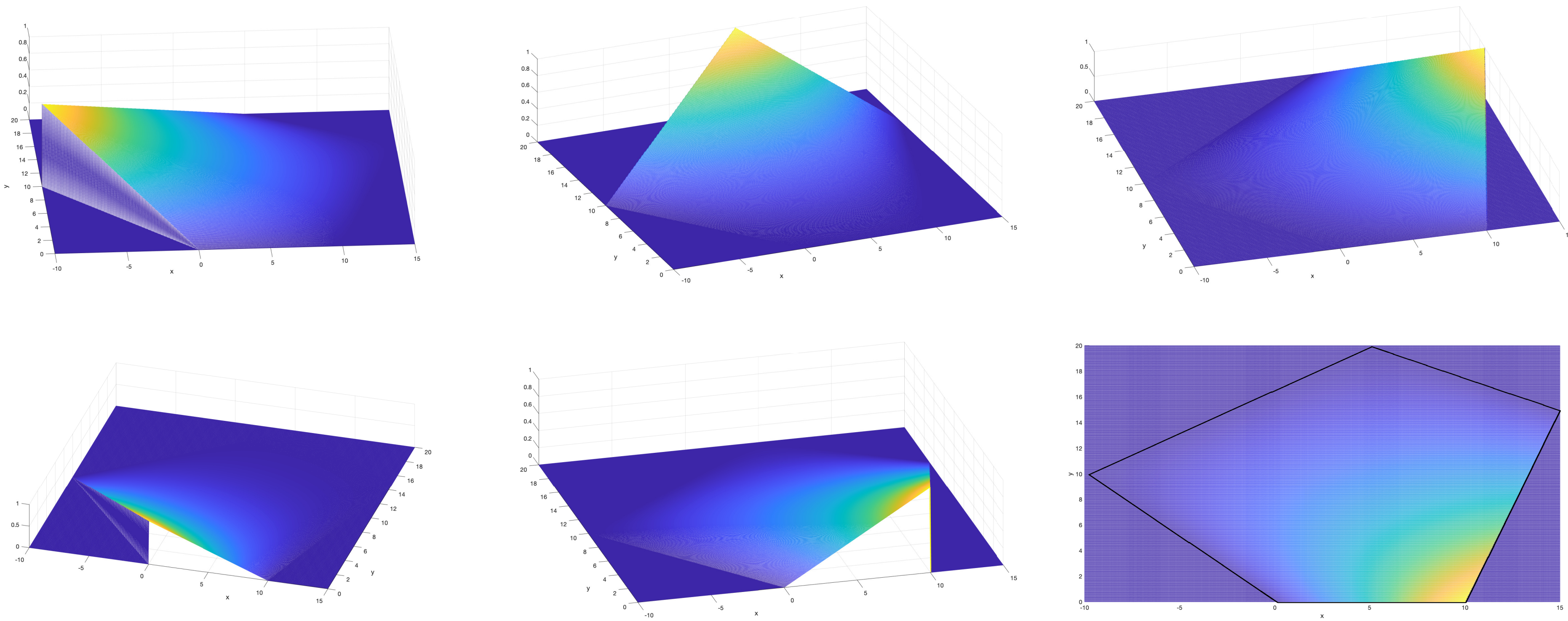
Edge





Basis Functions of $\mathcal{DS}_1(E_5)$

$$\mathcal{DS}_r(E_N) = \left\{\varphi \in \mathcal{DS}_{N-2}(E_N) : \varphi|_{e_i} \in \mathbb{P}_r(e_i) \text{ for all edges } e_i \text{ of } E_N\right\}$$



Remark. Lack of symmetry makes it difficult to define supplements!

Remark. These appear to be barycentric coordinates (but we do not prove they are nonnegative).

De Rham Complex. The image of one map is the kernel of the next.

$$\boxed{\mathbb{R} \hookrightarrow \mathcal{DS}_{r+1}(E_N) \xrightarrow{\text{curl}} \mathbf{V}_r^s(E_N) \xrightarrow{\text{div}} \mathbb{P}_s(E_N) \longrightarrow 0}$$

Recall

- $\mathbb{P}_r^2 = \operatorname{curl} \mathbb{P}_{r+1} \oplus \mathbf{x}\mathbb{P}_{r-1}$
- $\nabla \cdot : \mathbf{x}\mathbb{P}_s \to \mathbb{P}_s$ is one-to-one and onto

Decomposition into direct finite elements.

Reduced $H(\text{div})$-approximating: $s = r - 1$, $r = 1, 2, \dots$

$$\begin{aligned} \mathbf{V}_r^{r-1}(E_N) &= \operatorname{curl} \mathcal{DS}_{r+1}(E_N) \oplus \mathbf{x}\mathbb{P}_{r-1} \\ &= \mathbb{P}_r^2(E_N) \oplus \mathbb{S}_r^{\mathbf{V}}(E_N) \end{aligned}$$

Full $H(\text{div})$-approximating: $s = r$, $r = 0, 1, \dots$

$$\begin{aligned} \mathbf{V}_r^{r}(E_N) &= \operatorname{curl} \mathcal{DS}_{r+1}(E_N) \oplus \mathbf{x}\mathbb{P}_{r} \\ &= \mathbb{P}_r^2(E_N) \oplus \mathbf{x}\underbrace{\tilde{\mathbb{P}}_r}_{\text{homogeneous polynomials}} \oplus \mathbb{S}_r^{\mathbf{V}}(E_N) \end{aligned}$$

The supplemental (vector valued) functions are

$$\mathbb{S}_r^{\mathbf{V}}(E_N) = \operatorname{curl} \mathbb{S}_{r+1}^{\mathcal{DS}}(E_N)$$

Generalized Scott-Zhang Interpolation Operator. For $\mathcal{DS}_r = \text{span}\{\phi_j\}$,

$$\mathcal{I}_h^r v(\mathbf{x}) = \sum_i \phi_i(\mathbf{x}) \int_{K_i} \psi_i(\mathbf{y})\, v(\mathbf{y})\, dy \in \mathcal{DS}_r$$

where the dual basis is

$$\int_{K_i} \psi_i(\mathbf{x})\, \phi_j(\mathbf{x})\, dx = \delta_{ij} \qquad (K_i \text{ is an edge or } E)$$

Theorem (Optimal approx.). Assume $\mathcal{T}_h$ is uniformly shape regular and

$$1 \le p \le \infty \quad \text{and} \quad l > 1/p \ (l \ge 1 \text{ if } p = 1)$$

Then there exists $C = C(r, \sigma_*) > 0$, such that $\forall v \in W_p^l(E)$,

$$||v - \mathcal{I}_h^r v||_{W_p^m(E)} \le C\, h_E^{l-m} |v|_{W_p^l(E)}, \qquad 0 \le m \le l \le r+1$$

Moreover, there exists $C = C(r, \sigma_*) > 0$, such that for $v \in W_p^l(\Omega)$,

$$\Big(\sum_{E \in \mathcal{T}_h} ||v - \mathcal{I}_h^r v||^p_{W_p^m(E)} \Big)^{1/p} \le C\, h^{l-m}\, |v|_{W_p^l(\Omega)}, \qquad 0 \le m \le l \le r+1$$

Generalized Raviart-Thomas projection operator. For $\epsilon > 0$, construct

$$\pi : H(\mathrm{div};\Omega) \cap (L^{2+\epsilon}(\Omega))^2 \to \mathbf{V}_r^s, \qquad s = r-1, r \ (s \geq 0)$$

- π is pieced together from locally defined operators π_E
- $\pi_E \mathbf{v}$ is defined in terms of the DoFs
- $\mathcal{P}_{W_s} \nabla \cdot \mathbf{v} = \nabla \cdot \pi \mathbf{v}$, $\mathcal{P}_{W_s}$ is L^2-projection onto $W_s = \nabla \cdot \mathbf{V}_r^s = \mathbb{P}_s$

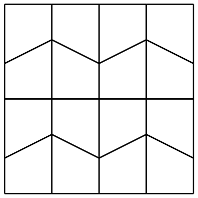
Theorem (Optimal approximation). For $s = r-1, r$ ($s \geq 0$), define

$$\begin{aligned}
\|\mathbf{v} - \pi\mathbf{v}\|_{0,\Omega} &\leq C\,\|\mathbf{v}\|_{k,\Omega}\, h^k & k &= 1, \dots, r+1 \\
\|\nabla \cdot (\mathbf{v} - \pi\mathbf{v})\|_{0,\Omega} &\leq C\,\|\nabla \cdot \mathbf{v}\|_{k,\Omega}\, h^k & k &= 0, 1, \dots, s+1 \\
\|p - \mathcal{P}_{W_s} p\|_{0,\Omega} &\leq C\,\|p\|_{k,\Omega}\, h^k & k &= 0, 1, \dots, s+1
\end{aligned}$$

Moreover, the discrete inf-sup condition holds for some $\gamma > 0$:

$$\sup_{\mathbf{v}_h \in \mathbf{V}_r^s} \frac{(w_h, \nabla \cdot \mathbf{v}_h)}{\|\mathbf{v}_h\|_{H(\mathrm{div})}} \geq \gamma\,\|w_h\|_{0,\Omega}, \qquad \forall w_h \in W_s = \mathbb{P}_s$$

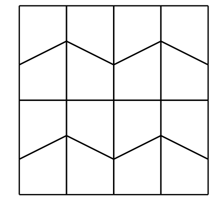
L^2-errors and convergence rates on trapezoidal meshes



n	$r=2$ error	rate	$r=3$ error	rate	$r=4$ error	rate	$r=5$ error	rate
				mapped $\mathbb{P}_{r,r}$				
8	3.329e-04	2.99	9.740e-06	3.99	2.382e-07	4.99	5.076e-09	5.99
12	9.888e-05	2.99	1.928e-06	3.99	3.142e-08	5.00	4.462e-10	6.00
16	4.176e-05	3.00	6.107e-07	4.00	7.459e-09	5.00	7.946e-11	6.00
24	1.238e-05	3.00	1.207e-07	4.00	9.827e-10	5.00	6.979e-12	6.00
		$\mathcal{O}(h^3)$		$\mathcal{O}(h^4)$		$\mathcal{O}(h^5)$		$\mathcal{O}(h^6)$
				mapped $\mathcal{S}_r$				
8	5.714e-04	2.92	4.844e-04	2.89	2.612e-05	3.72	2.005e-06	4.13
12	1.731e-04	2.94	1.482e-04	2.92	6.084e-06	3.59	3.884e-07	4.05
16	7.409e-05	2.95	6.383e-05	2.93	2.265e-06	3.43	1.234e-07	3.99
24	2.254e-05	2.94	1.963e-05	2.91	5.984e-07	3.28	2.516e-08	3.92
48	3.127e-06	2.82	2.825e-06	2.76	6.875e-08	3.09	1.850e-09	3.71
64	1.440e-06	2.70	1.332e-06	2.61	2.862e-08	3.05	6.644e-10	3.56
		$\mathcal{O}(h^?)$		$\mathcal{O}(h^?)$		$\mathcal{O}(h^?)$		$\mathcal{O}(h^?)$
				direct $\mathcal{DS}_r$				
8	3.492e-04	3.00	3.897e-05	4.07	2.187e-06	5.00	8.896e-08	5.96
12	1.036e-04	3.00	7.457e-06	4.08	2.889e-07	4.99	7.870e-09	5.98
16	4.373e-05	3.00	2.313e-06	4.07	6.868e-08	4.99	1.404e-09	5.99
24	1.296e-05	3.00	4.469e-07	4.05	9.058e-09	5.00	1.235e-10	6.00
		$\mathcal{O}(h^3)$		$\mathcal{O}(h^4)$		$\mathcal{O}(h^5)$		$\mathcal{O}(h^6)$

Convergence for $\mathbb{P}_{r,r}$, $\mathcal{S}_r$, and $\mathcal{DS}_r$ on Trapezoids — 2

H^1-seminorm errors and conv. rates on trapezoidal meshes

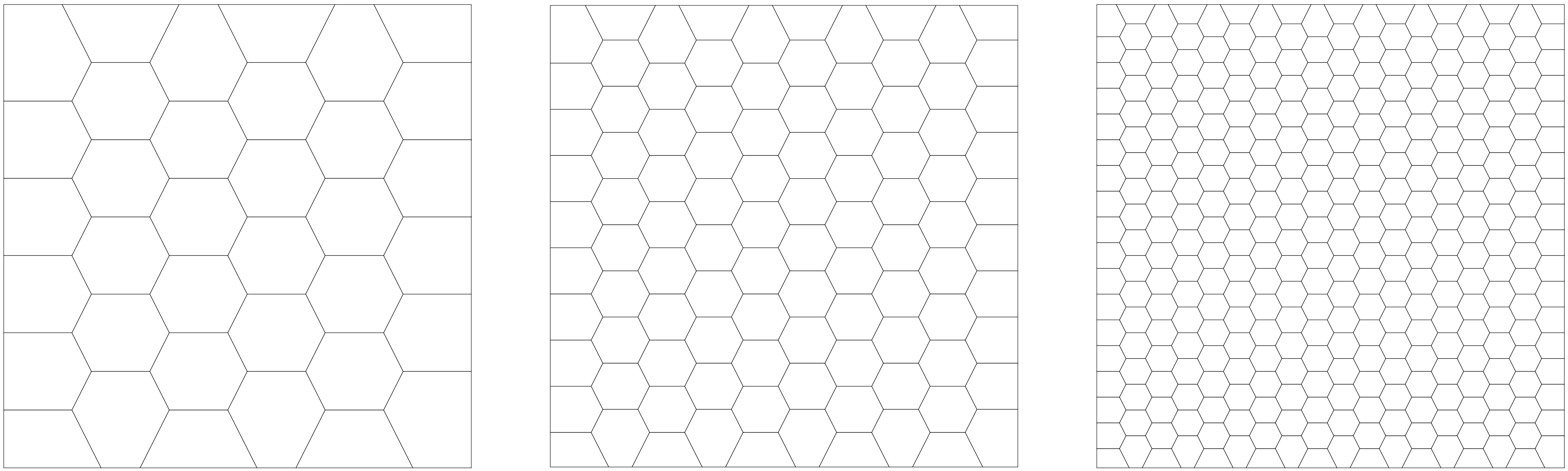


n	$r=2$ error	rate	$r=3$ error	rate	$r=4$ error	rate	$r=5$ error	rate
			$\mathbb{P}_{r,r}$ on $\mathcal{T}_h^2$ meshes					
8	1.734e-02	2.00	7.206e-04	2.99	2.310e-05	3.99	6.083e-07	4.99
12	7.710e-03	2.00	2.139e-04	3.00	4.570e-06	4.00	8.021e-08	5.00
16	4.337e-03	2.00	9.027e-05	3.00	1.447e-06	4.00	1.904e-08	5.00
24	1.928e-03	2.00	2.676e-05	3.00	2.859e-07	4.00	2.509e-09	5.00
		$\mathcal{O}(h^2)$		$\mathcal{O}(h^3)$		$\mathcal{O}(h^4)$		$\mathcal{O}(h^5)$
			$\mathcal{S}_r$ on $\mathcal{T}_h^2$ meshes					
8	2.413e-02	1.94	1.834e-02	1.90	1.818e-03	2.65	1.537e-04	3.18
12	1.105e-02	1.93	8.572e-03	1.88	6.582e-04	2.51	4.483e-05	3.04
16	6.432e-03	1.88	5.091e-03	1.81	3.345e-04	2.35	1.945e-05	2.90
24	3.104e-03	1.80	2.560e-03	1.70	1.360e-04	2.22	6.370e-06	2.75
48	1.043e-03	1.50	9.409e-04	1.37	3.190e-05	2.07	1.140e-06	2.41
64	7.097e-04	1.34	6.602e-04	1.23	1.776e-05	2.03	5.953e-07	2.26
		$\mathcal{O}(h^?)$		$\mathcal{O}(h^?)$		$\mathcal{O}(h^?)$		$\mathcal{O}(h^?)$
			$\mathcal{DS}_r$ on $\mathcal{T}_h^2$ meshes					
8	1.836e-02	2.01	2.517e-03	3.02	1.625e-04	3.99	7.384e-06	4.99
12	8.143e-03	2.00	7.400e-04	3.02	3.216e-05	4.00	9.757e-07	4.99
16	4.577e-03	2.00	3.109e-04	3.01	1.018e-05	4.00	2.318e-07	5.00
24	2.033e-03	2.00	9.170e-05	3.01	2.012e-06	4.00	3.056e-08	5.00
		$\mathcal{O}(h^2)$		$\mathcal{O}(h^3)$		$\mathcal{O}(h^4)$		$\mathcal{O}(h^5)$

Manufactured solution. $u(x,y) = \sin(\pi x)\sin(\pi y)$ solving

$$-\Delta u = f, \quad 0 < x < 1,\ 0 < y < 1$$

Meshes. $N = 6$ mostly, but some $N = 4, 5$



36 elements   100 elements   324 elements

Meshes from PolyMesher (Talischi, Paulino, Pereira & Menezes 2012)

Errors and convergence rates for direct serendipity spaces

n	h	$r=2$		$r=3$		$r=4$		$r=5$	
				L^2-norm					
4	0.763	3.947E-2	—	1.461E-2	—	2.487E-3	—	3.882E-4	—
36	0.254	1.017E-3	3.32	7.804E-5	4.52	5.508E-6	5.37	2.740E-7	6.52
100	0.153	1.991E-4	3.16	8.639E-6	4.25	3.549E-7	5.35	9.891E-9	6.48
196	0.109	6.960E-5	3.12	2.129E-6	4.14	5.921E-8	5.31	1.152E-9	6.33
324	0.085	3.199E-5	3.09	7.595E-7	4.09	1.568E-8	5.28	2.384E-10	6.24
			$\mathcal{O}(h^3)$		$\mathcal{O}(h^4)$		$\mathcal{O}(h^5)$		$\mathcal{O}(h^6)$
				H^1-seminorm					
4	0.763	1.066E-1	—	4.639E-2	—	1.032E-2	—	1.922E-3	—
36	0.254	9.801E-3	2.24	9.334E-4	3.36	8.123E-5	4.35	4.296E-6	5.50
100	0.153	3.223E-3	2.16	1.826E-4	3.15	8.844E-6	4.33	2.669E-7	5.41
196	0.109	1.575E-3	2.12	6.441E-5	3.08	2.083E-6	4.29	4.383E-8	5.36
324	0.085	9.285E-4	2.10	2.985E-5	3.05	7.138E-7	4.25	1.150E-8	5.32
			$\mathcal{O}(h^2)$		$\mathcal{O}(h^3)$		$\mathcal{O}(h^4)$		$\mathcal{O}(h^5)$

Convergence Study for $\mathbf{V}_r^s$ on Polygons

Errors and convergence rates for direct mixed spaces

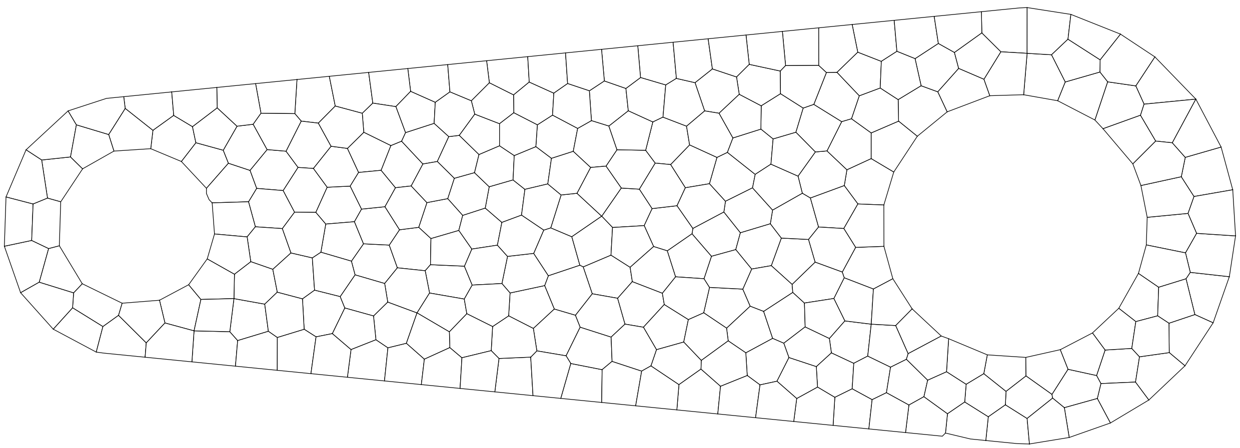
n	h	$\|\|p - p_h\|\|$ error	rate	$\|\|\mathbf{u} - \mathbf{u}_h\|\|$ error	rate	$\|\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|\|$ error	rate
$r = s = 1$, full $H(\text{div})$-approximation							
36	0.254	2.452E-02	2.05	8.125E-03	2.42	2.450E-02	2.04
100	0.153	8.641E-03	2.04	2.403E-03	2.37	8.639E-03	2.04
196	0.109	4.363E-03	2.03	1.094E-03	2.33	4.363E-03	2.03
324	0.085	2.624E-03	2.02	6.133E-04	2.29	2.624E-03	2.02
			$\mathcal{O}(h^2)$		$\mathcal{O}(h^2)$		$\mathcal{O}(h^2)$
$r = 1$, $s = 0$, reduced $H(\text{div})$-approximation							
36	0.254	2.296E-01	1.22	5.183E-02	2.09	2.152E-01	1.04
100	0.153	1.308E-01	1.08	1.820E-02	2.04	1.277E-01	1.02
196	0.109	9.196E-02	1.04	9.199E-03	2.03	9.084E-02	1.01
324	0.085	7.104E-02	1.02	5.539E-03	2.02	7.051E-02	1.01
			$\mathcal{O}(h^1)$		$\mathcal{O}(h^2)$		$\mathcal{O}(h^1)$
$r = s = 2$, full $H(\text{div})$-approximation							
36	0.254	1.853E-03	3.14	4.217E-04	3.37	1.853E-03	3.14
100	0.153	3.858E-04	3.06	7.535E-05	3.37	3.858E-04	3.06
196	0.109	1.385E-04	3.04	2.420E-05	3.37	1.385E-04	3.04
324	0.085	6.464E-05	3.03	1.038E-05	3.37	6.464E-05	3.03
			$\mathcal{O}(h^3)$		$\mathcal{O}(h^3)$		$\mathcal{O}(h^3)$
$r = 2$, $s = 1$, reduced $H(\text{div})$-approximation							
36	0.254	2.452E-02	2.05	2.393E-03	3.08	2.450E-02	2.04
100	0.153	8.640E-03	2.04	5.053E-04	3.04	8.639E-03	2.04
196	0.109	4.363E-03	2.03	1.825E-04	3.03	4.363E-03	2.03
324	0.085	2.624E-03	2.02	8.545E-05	3.02	2.624E-03	2.02
			$\mathcal{O}(h^2)$		$\mathcal{O}(h^3)$		$\mathcal{O}(h^2)$

Manufactured solution. $u(x, y) = \sin(\pi x)\sin(\pi y)$ solving

$$-\Delta u = f$$

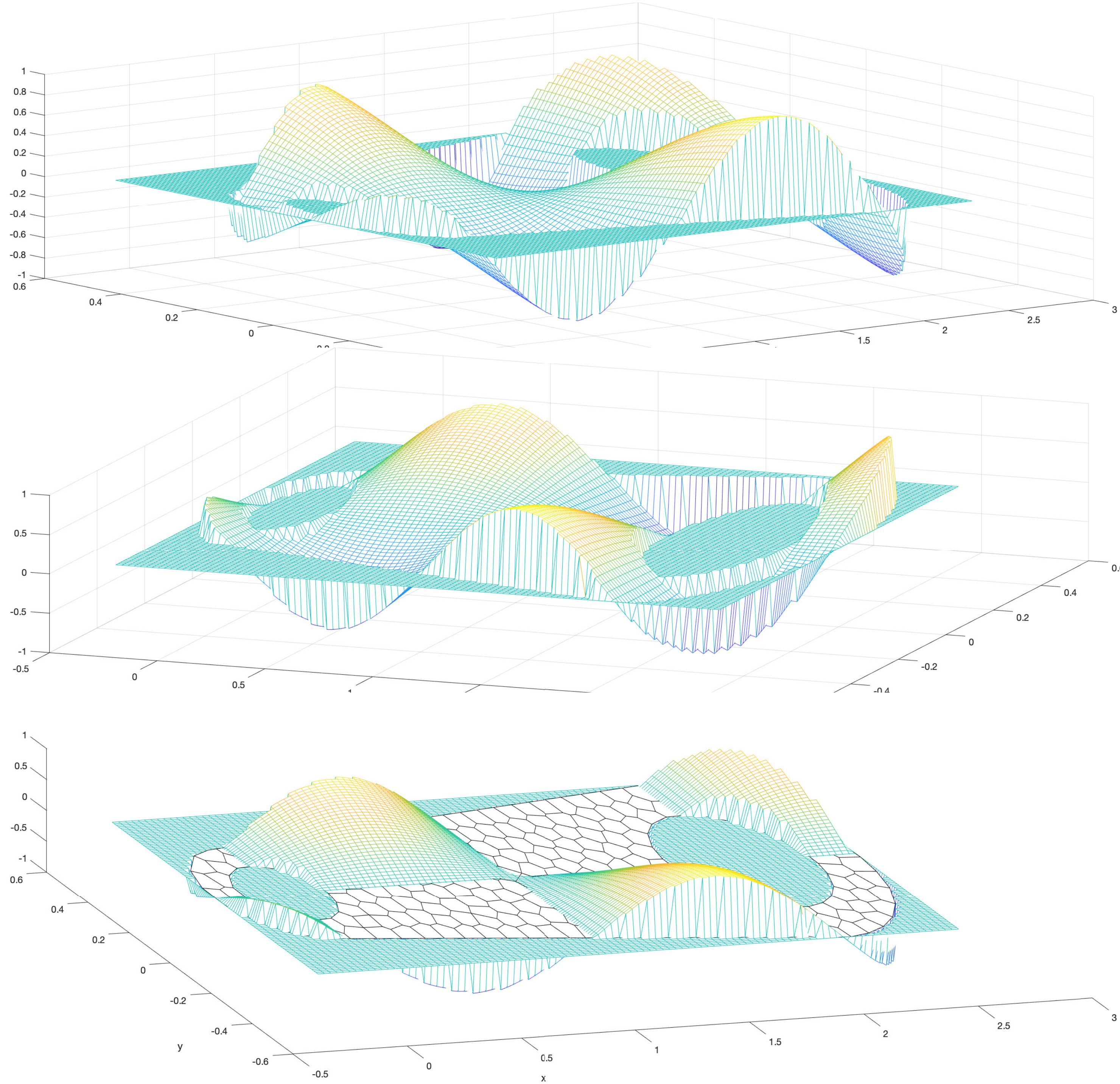
Solve with $r = 2$.

Mesh. 250 elements, 506 vertices, N from 4 to 8



Mesh from PolyMesher (Talischi, Paulino, Pereira & Menezes 2012)

Wrench Example — 2



Summary and Conclusions — 1

1. Conforming finite elements on polygons are important in many areas.
   - Solving PDEs in certain applications.
   - General interpolation and approximation of functions.
   - Visualization.
2. Direct spaces (polynomials plus supplements) offer advantages
   - Quadrilaterals: no accuracy loss due to reference element mapping.
   - Polygons: require no reference element.
3. Direct serendipity finite elements developed for convex polygons.
   - H^1-conforming and fully constructive. Keys to the construction:
     - Cannot define spaces for $r = 1$, then $r = 2$, then $r = 3$, etc.
     - $r \geq N - 2$: cell and vertex basis functions straightforward.
     - Edge basis functions require supplements using functions $R_{i,j}(\mathbf{x})$ that are ± 1 on nonadjacent edges e_i and e_j.
     - $r < N - 2$: delicate, so take $\mathcal{DS}_r(E_N) \subset \mathcal{DS}_{N-2}(E_N)$.
   - Minimal DoFs and approximate optimally on shape regular meshes.

4. Direct mixed finite elements developed for convex polygons.
   - $H(\mathrm{div})$-conforming direct and fully constructive.
   - Arise from a de Rham complex using FEEC.
   - Full and reduced $H(\mathrm{div})$-approximating spaces.
   - Minimal DoFs and approximate optimally on shape regular meshes.
5. Extensions to 3-D polytopes.
   - We have defined direct serendipity and some mixed spaces on hexahedra.
   - The restriction of the $\mathcal{DS}_r$ on hexahedra to each face f fall into the $\mathcal{DS}_r(f)$ defined on quadrilateral.
   - Direct mixed spaces are difficult because:
     - The de Rham complex is longer
     $$\mathbb{R} \hookrightarrow H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0$$
     - We need to define a space of vectors $\mathbf{X}$ so that
$$\mathbb{R} \hookrightarrow \mathcal{DS}_{r+2}(E) \xrightarrow{\text{grad}} \mathbf{W}_{r+1}(E) = \text{grad}\,\mathcal{DS}_{r+2}(E) \oplus \mathbf{X}$$
$$\xrightarrow{\text{curl}} \mathbf{V}_r^s(E) = \text{curl}\,\mathbf{X} \oplus \mathbf{x}\mathbb{P}_s(E) \xrightarrow{\text{div}} \mathbb{P}_s(E) \longrightarrow 0$$

References

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