Implementation of Direct Serendipity and Mixed Finite Elements on Convex Polygons

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1. Construction of Direct Serendipity Elements $\mathcal{DS}_r(E_N)$

$$\mathcal{DS}_r(E_N) = \mathbb{P}_r(E_N) \oplus \mathbb{S}_r^{\mathcal{DS}}(E_N)$$

polynomials plus supplements

The minimal DoFs needed for H^1-Conformity ($N \le 3, r \le 1$) implies

$$\mathcal{DS}_r(E_N) = \mathbb{P}_r(E_N) \oplus \mathbb{S}_r^{\mathcal{DS}}(E_N)$$

where

$$\mathbb{P}_r(E_N) \cap \mathbb{S}_r^{\mathcal{DS}}(E_N) = \emptyset$$

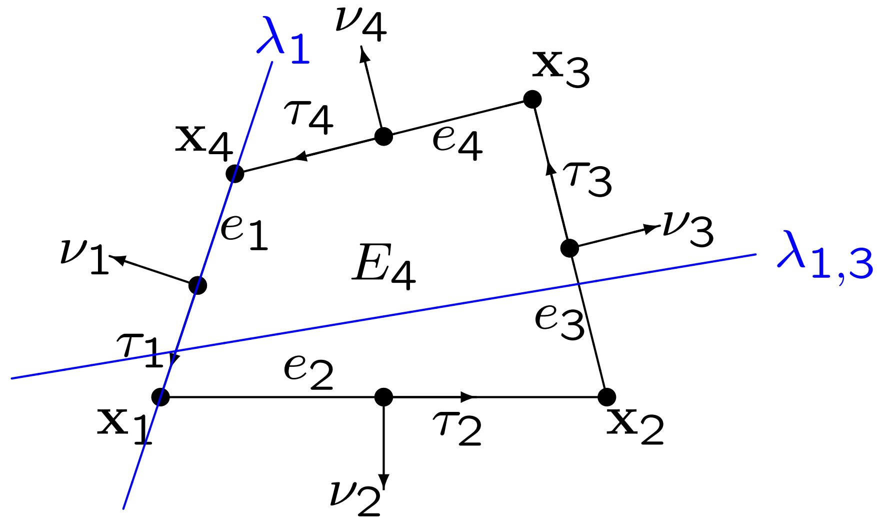
$$\dim \mathbb{S}_r^{\mathcal{DS}}(E_N) = \begin{cases} \frac{1}{2}N(N-3), & r \ge N-2 \\ Nr - \frac{1}{2}(r+2)(r+1) < \frac{1}{2}N(N-3), & r < N-2 \end{cases}$$

Counterintuitive observation. The case $r \ge N-2$ is easier!

- For $r < N-2$: We will define $\mathcal{DS}_r(E_N) \subset \mathcal{DS}_{N-2}(E_N)$.
- For $r \ge N-2$: The number of nonadjacent edge pairs is

$$\frac{1}{2}N(N-3) = \dim \mathbb{S}_r^{\mathcal{DS}}(E_N)$$

Supplemental basis functions are associated to pairs of edges!



Use CCW Ordering
(and mod N as needed).

Linear polynomial λ_i for edge e_i. Define

$$\lambda_i(\mathbf{x}) = -(\mathbf{x} - \mathbf{x}_i) \cdot \nu_i \quad \propto \quad \text{distance of } \mathbf{x} \text{ to the line through edge } e_i$$

$$\Longrightarrow \lambda_i\Big|_{e_i} = 0 \quad \text{(zero line contains } e_i\text{)}$$

$$\lambda_i > 0 \quad \text{on the interior of } E_N$$

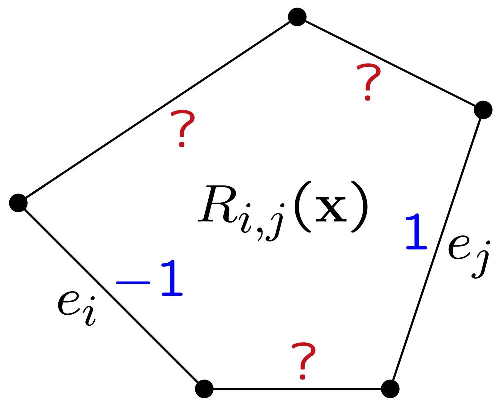
Not barycentric coordinates!

Linear polynomial $\lambda_{i,j}$ for edges e_i and e_j. Choose *any* linear polynomial $\lambda_{i,j}$ with zero line joining e_i and e_j. (Connect the midpoints?)

Special "Rational" Functions

When e_i and e_j are *not* adjacent and $i < j \pmod N$, let

$$R_{i,j}(\mathbf{x}) = \frac{\lambda_i(\mathbf{x}) - \lambda_j(\mathbf{x})}{\lambda_i(\mathbf{x}) + \lambda_j(\mathbf{x})} \quad \Longrightarrow \quad \begin{cases} R_{i,j}(\mathbf{x})|_{e_i} = -1 \\ R_{i,j}(\mathbf{x})|_{e_j} = 1 \end{cases}$$



Remark. One could use any (non rational) function $R_{i.j}$ with the required properties. How to define?

Theorem. The finite element

$$\mathcal{DS}_r(E_N) = \mathbb{P}_r(E_N) \oplus \mathbb{S}_r^{\mathcal{DS}}(E_N)$$

$$\mathbb{S}_r^{\mathcal{DS}}(E_N) = \{\omega_{i,j} : i,j = 1,\ldots,N,\ i,j \text{ nonadjacent}\}$$

$$\omega_{i,j} = \Big(\prod_{m \neq i,j} \lambda_m\Big)\lambda_{i,j}^{r-N+2} R_{i,j}$$

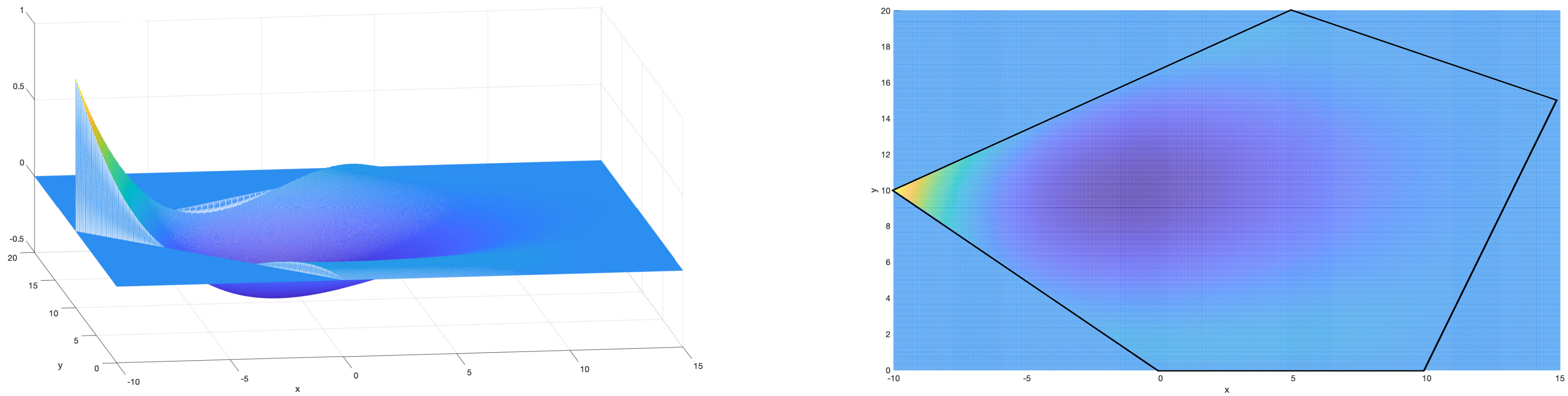
(with nodal DoFs) is well defined (i.e., unisolvent).

Moreover, it has the minimal number of DoFs needed to

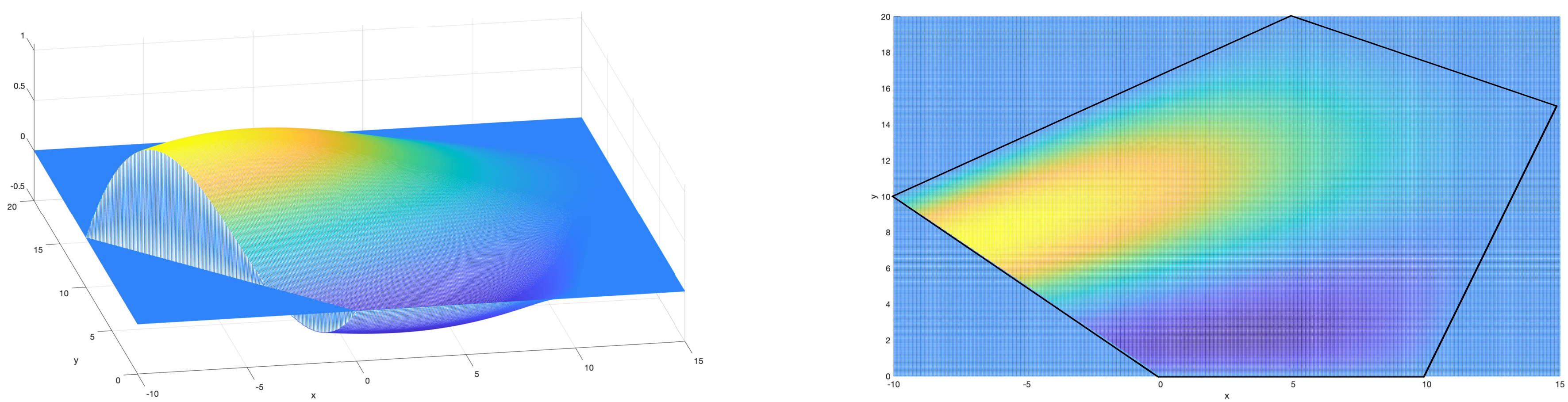
- contain $\mathbb{P}_r$
- be H^1 conforming

Basis Functions of $\mathcal{DS}_3(E_5)$

Vertex



Edge



Definition of $\mathcal{DS}_r(E_N)$, $r < N-2$

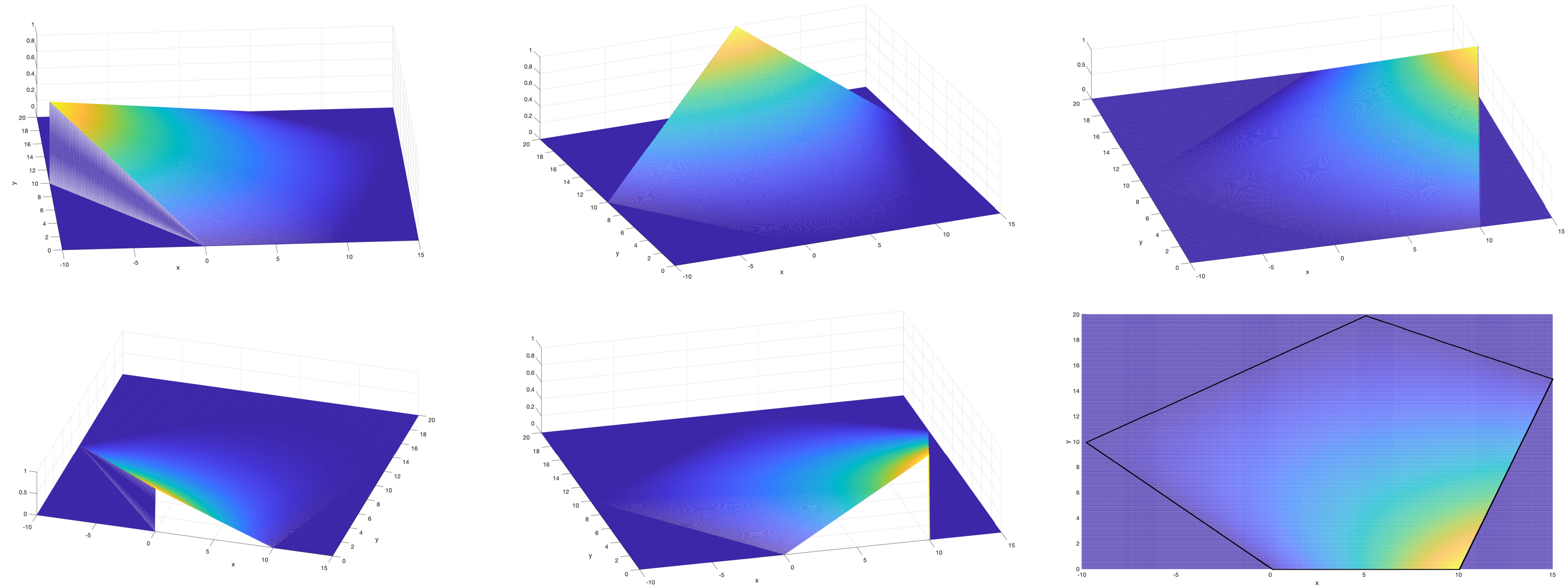
$$\mathcal{DS}_r(E_N) = \left\{\varphi \in \mathcal{DS}_{N-2}(E_N) : \varphi|_{e_i} \in \mathbb{P}_r(e_i) \text{ for all edges } e_i \text{ of } E_N\right\}$$

Theorem. The finite element $\mathcal{DS}_r(E_N)$ with nodal DoFs is well defined (i.e., unisolvent) when $r < N-2$. Moreover,

$$\mathcal{DS}_r(E_N) = \mathbb{P}_r(E_N) \oplus \mathbb{S}_r^{\mathcal{DS}}(E_N) \ \subset \ \mathcal{DS}_{N-2}(E_N)$$

for some supplemental space of minimal dimension needed for H^1-conformity.

Basis Functions of $\mathcal{DS}_1(E_5)$



Remark. These appear to be barycentric coordinates (but we do not prove they are nonnegative).

2. Direct Mixed Elements

$$\mathbf{V}_r(E_N) = \mathbb{P}_r^2(E_N) \oplus \mathbb{S}_r^{\mathbf{V}}(E_N)$$

Vector valued polynomials plus supplements

Structure of $\mathbf{V}_r^s(E_N)$

$$\mathbb{R} \hookrightarrow \mathcal{DS}_{r+1}(E_N) \xrightarrow{\text{curl}} \mathbf{V}_r^s(E_N) \xrightarrow{\text{div}} \mathbb{P}_s(E_N) \longrightarrow 0$$

Recall

- $\mathbb{P}_r^2 = \operatorname{curl} \mathbb{P}_{r+1} \oplus \mathbf{x}\mathbb{P}_{r-1}$
- $\nabla\cdot : \mathbf{x}\mathbb{P}_s \to \mathbb{P}_s$ is one-to-one and onto

Decomposition into direct finite elements

Reduced H(div)-approximating: $s = r - 1$, $r = 1, 2, \ldots$

$$\begin{aligned} \mathbf{V}_r^{r-1}(E_N) &= \operatorname{curl} \mathcal{DS}_{r+1}(E_N) \oplus \mathbf{x}\mathbb{P}_{r-1} \\ &= \mathbb{P}_r^2(E_N) \oplus \mathbb{S}_r^{\mathbf{V}}(E_N) \end{aligned}$$

Full H(div)-approximating: $s = r$, $r = 0, 1, \ldots$

$$\begin{aligned} \mathbf{V}_r^{r}(E_N) &= \operatorname{curl} \mathcal{DS}_{r+1}(E_N) \oplus \mathbf{x}\mathbb{P}_{r} \\ &= \mathbb{P}_r^2(E_N) \oplus \mathbf{x} \underbrace{\tilde{\mathbb{P}}_r}_{\text{homogeneous polynomials}} \oplus \mathbb{S}_r^{\mathbf{V}}(E_N) \end{aligned}$$

The supplemental (vector valued) functions are

$$\mathbb{S}_r^{\mathbf{V}}(E_N) = \operatorname{curl} \mathbb{S}_{r+1}^{\mathcal{DS}}(E_N)$$

3. Numerical Results

Approximation Properties of $\mathcal{DS}_r(E_4)$ and $\mathbf{V}_r^s(E_4)$

Theorem (Optimal approx.). Assume $\mathcal{T}_h$ is uniformly shape regular and

$$1 \le p \le \infty \quad \text{and} \quad l > 1/p \ (l \ge 1 \text{ if } p = 1)$$

Then for $v \in W_p^l(\Omega)$,

$$\Big(\sum_{E \in \mathcal{T}_h} \|v - \mathcal{I}_h^r v\|_{W_p^m(E)}^p \Big)^{1/p} \le C\, h^{l-m}\, |v|_{W_p^l(\Omega)}, \qquad 0 \le m \le l \le r+1$$

Theorem (Optimal approximation). For $s = r-1, r$ $(s \ge 0)$,

$$\begin{aligned} \|\mathbf{v} - \pi\mathbf{v}\|_{0,\Omega} &\le C\, \|\mathbf{v}\|_{k,\Omega}\, h^k & k &= 1, \dots, r+1 \\ \|\nabla \cdot (\mathbf{v} - \pi\mathbf{v})\|_{0,\Omega} &\le C\, \|\nabla \cdot \mathbf{v}\|_{k,\Omega}\, h^k & k &= 0, 1, \dots, s+1 \\ \|p - \mathcal{P}_{W_s} p\|_{0,\Omega} &\le C\, \|p\|_{k,\Omega}\, h^k & k &= 0, 1, \dots, s+1 \end{aligned}$$

Moreover, the discrete inf-sup condition holds for some $\gamma > 0$:

$$\sup_{\mathbf{v}_h \in \mathbf{V}_r^s} \frac{(w_h, \nabla \cdot \mathbf{v}_h)}{\|\mathbf{v}_h\|_{H(\mathrm{div})}} \ge \gamma\, \|w_h\|_{0,\Omega}, \qquad \forall w_h \in W_s = \mathbb{P}_s$$

Remark. Our results should extend to polygons.

Manufactured solution. $u(x, y) = \sin(\pi x)\sin(\pi y)$ solving

$$-\Delta u = f, \quad 0 < x < 1,\ 0 < y < 1.$$

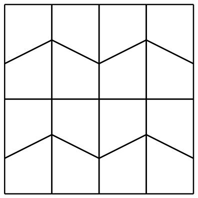
Weak form for testing $\mathcal{DS}_r$. Find $p \in H_0^1(\Omega)$ such that

$$(\nabla p, \nabla q) = (f, q), \quad \forall q \in H_0^1(\Omega).$$

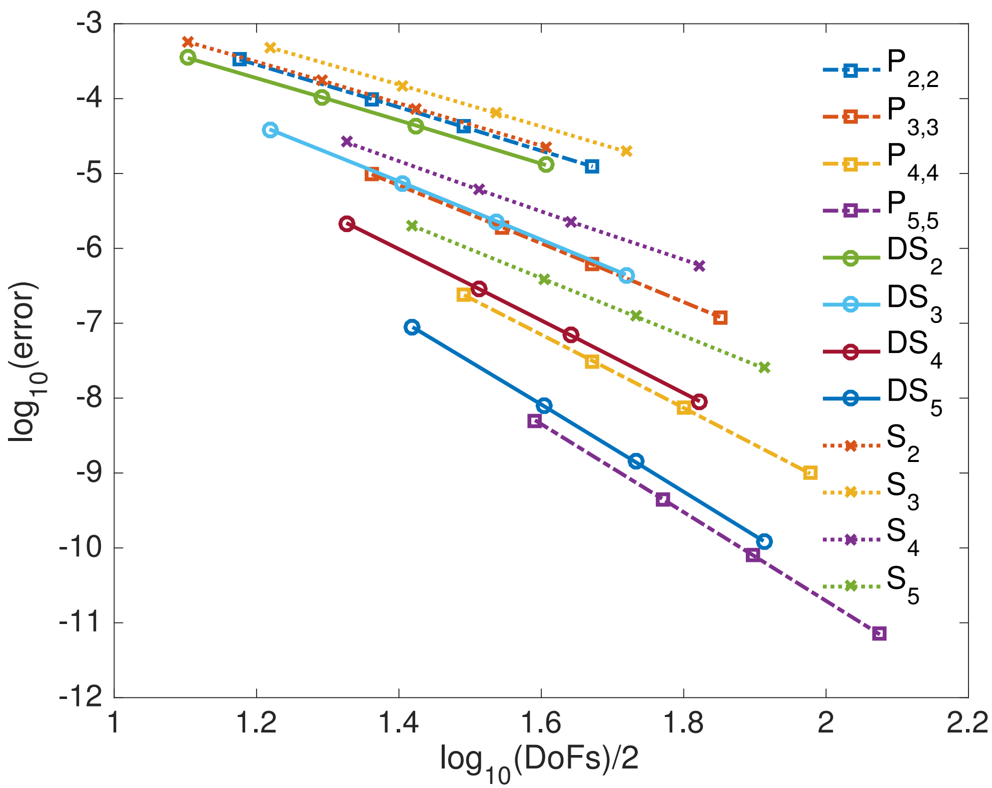
Weak form for testing $\mathbf{V}_r^s$. Find $\mathbf{u} \in H(\text{div}; \Omega)$ and $p \in L^2(\Omega)$ such that

$$\begin{aligned} (\mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) &= 0, & \forall \mathbf{v} \in H(\text{div}; \Omega), \\ (\nabla \cdot \mathbf{u}, w) &= (f, w), & \forall w \in L^2(\Omega). \end{aligned}$$

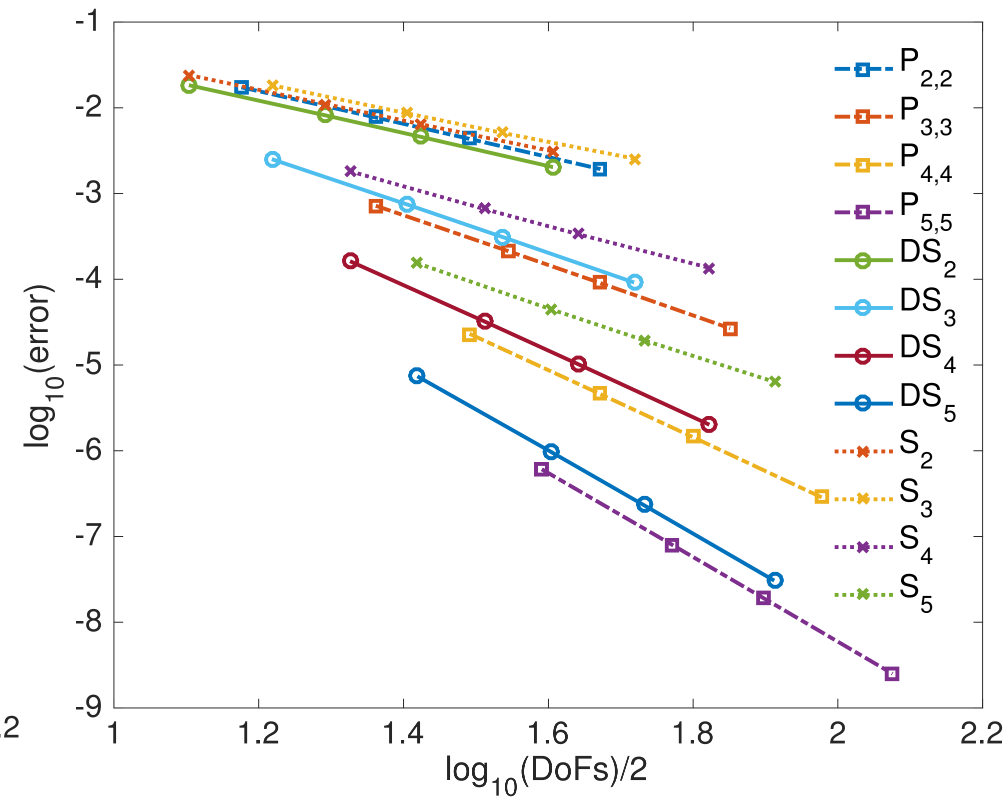
L^2-errors and convergence rates on trapezoidal meshes



n	$r=2$ error	rate	$r=3$ error	rate	$r=4$ error	rate	$r=5$ error	rate
	mapped $\mathbb{P}_{r,r}$							
64	3.329e-04	2.99	9.740e-06	3.99	2.382e-07	4.99	5.076e-09	5.99
144	9.888e-05	2.99	1.928e-06	3.99	3.142e-08	5.00	4.462e-10	6.00
256	4.176e-05	3.00	6.107e-07	4.00	7.459e-09	5.00	7.946e-11	6.00
576	1.238e-05	3.00	1.207e-07	4.00	9.827e-10	5.00	6.979e-12	6.00
		$\mathcal{O}(h^3)$		$\mathcal{O}(h^4)$		$\mathcal{O}(h^5)$		$\mathcal{O}(h^6)$
	mapped $\mathcal{S}_r$							
64	5.714e-04	2.92	4.844e-04	2.89	2.612e-05	3.72	2.005e-06	4.13
144	1.731e-04	2.94	1.482e-04	2.92	6.084e-06	3.59	3.884e-07	4.05
256	7.409e-05	2.95	6.383e-05	2.93	2.265e-06	3.43	1.234e-07	3.99
576	2.254e-05	2.94	1.963e-05	2.91	5.984e-07	3.28	2.516e-08	3.92
		$\mathcal{O}(h^?)$		$\mathcal{O}(h^?)$		$\mathcal{O}(h^?)$		$\mathcal{O}(h^?)$
	direct $\mathcal{DS}_r$							
64	3.492e-04	3.00	3.897e-05	4.07	2.187e-06	5.00	8.896e-08	5.96
144	1.036e-04	3.00	7.457e-06	4.08	2.889e-07	4.99	7.870e-09	5.98
256	4.373e-05	3.00	2.313e-06	4.07	6.868e-08	4.99	1.404e-09	5.99
576	1.296e-05	3.00	4.469e-07	4.05	9.058e-09	5.00	1.235e-10	6.00
		$\mathcal{O}(h^3)$		$\mathcal{O}(h^4)$		$\mathcal{O}(h^5)$		$\mathcal{O}(h^6)$



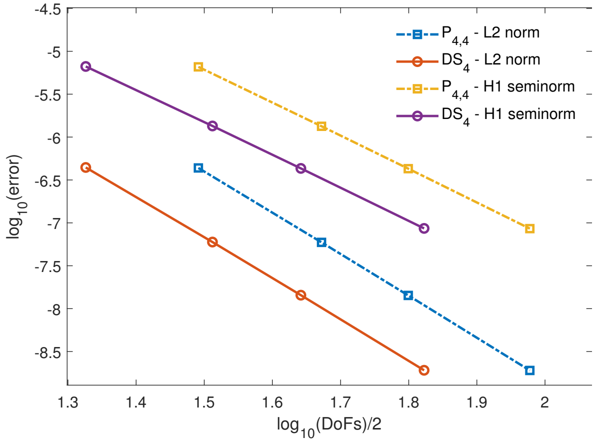
L^2-norm



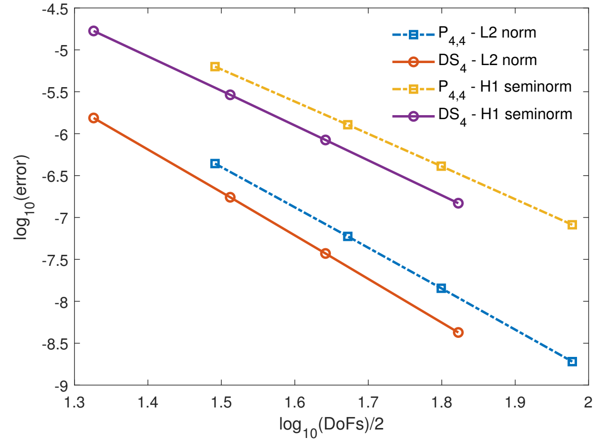
H^1-seminorm

Log scale L^2-norm (left) and H^1-seminorm (right) errors versus number of DoFs for $\mathbb{P}_{r,r}$, $\mathcal{DS}_r$, and $\mathcal{S}_r$ spaces on trapezoidal meshes.

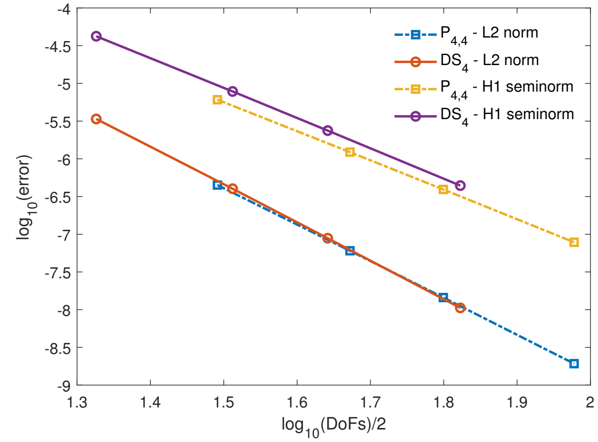
Suggestion. The $\mathbb{P}_{r,r}(E)$ finite elements are able to match many more 'cross-term' monomials than $\mathcal{DS}_r(E)$, up to the highest degree monomial $x^r y^r$.



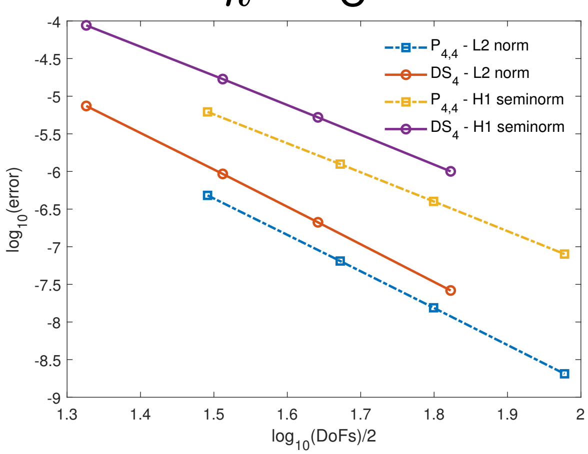
$k = 0$



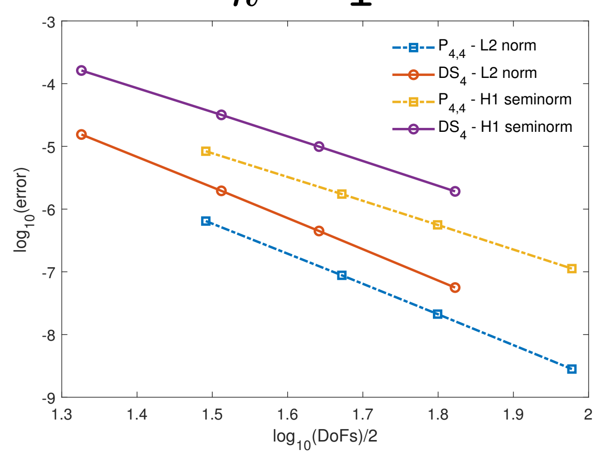
$k = 1$



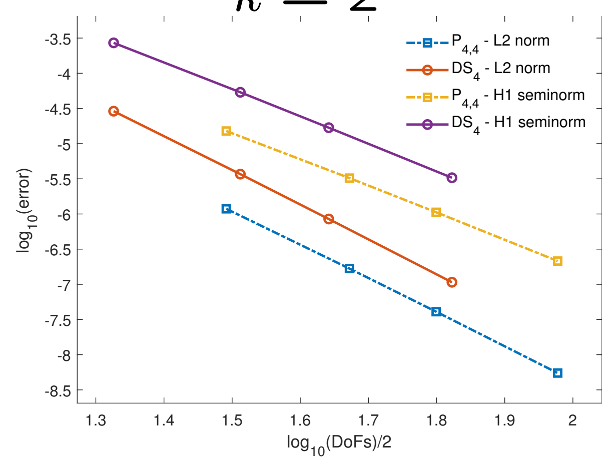
$k = 2$



$k = 3$

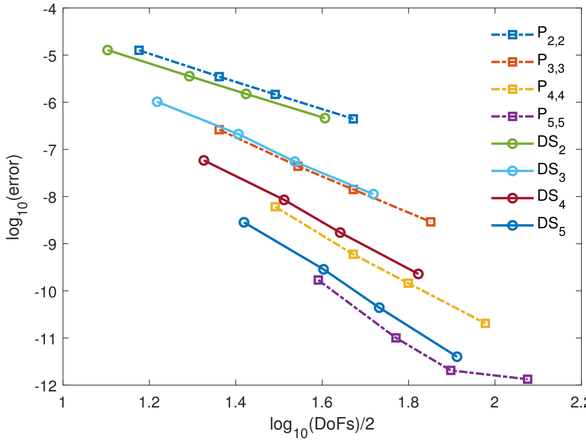


$k = 4$

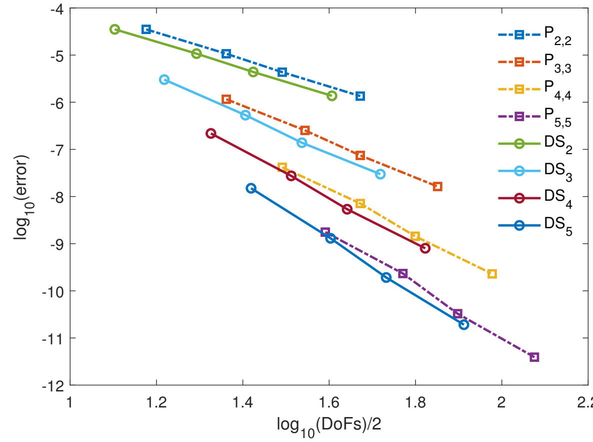


$k = 5$

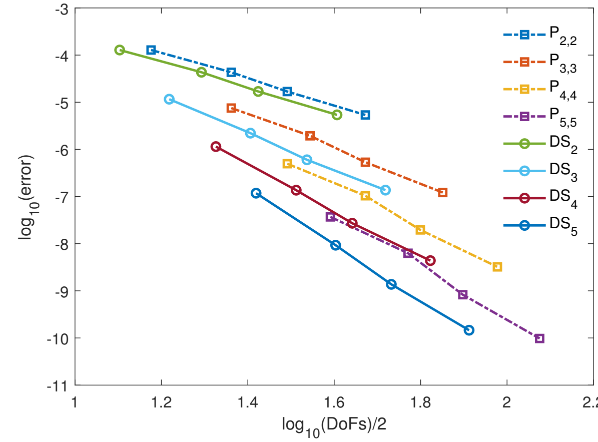
True solution $u(x,y) = x^5 y^k$. Log scale L^2-norm and H^1-seminorm errors versus the number of DoFs for tensor product $\mathbb{P}_{4,4}$ (blue and yellow dotted lines, squares) and $\mathcal{DS}_4$ (red and purple solid lines, circles) on trapezoidal meshes $\mathcal{T}_h^2$.



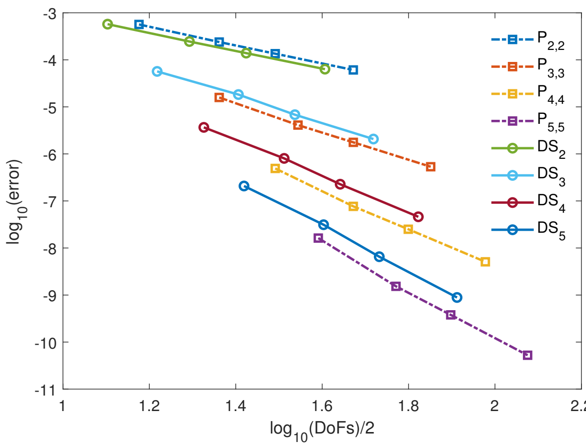
$k = 1$, L^2-norm



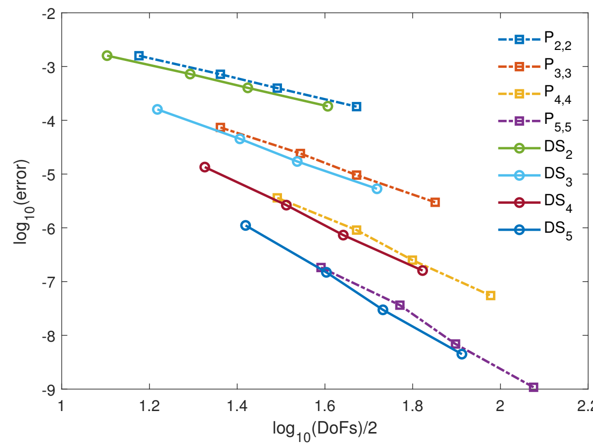
$k = 2$, L^2-norm



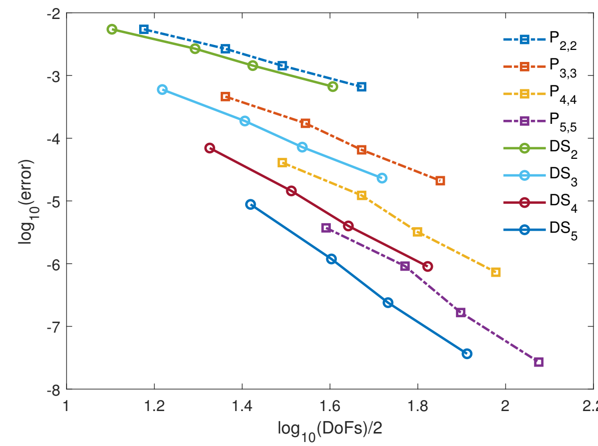
$k = 4$, L^2-norm



$k = 1$, H^1-seminorm



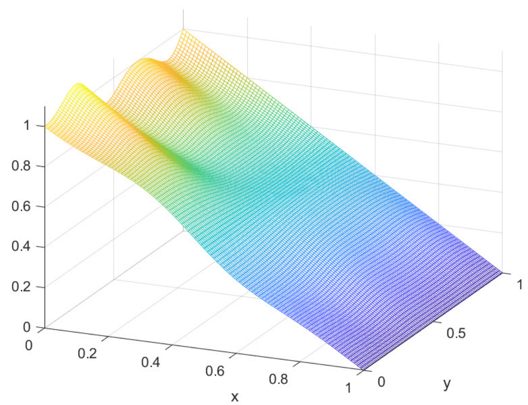
$k = 2$, H^1-seminorm



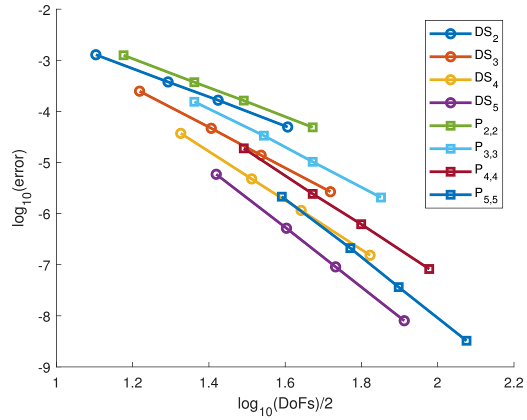
$k = 4$, H^1-seminorm

True solution $u(x,y) = \log(x + ky + 1)$. Log scale L^2-norm (top row) and H^1-seminorm errors (bottom row) versus the number of DoFs for tensor product $\mathbb{P}_{r,r}$ and $\mathcal{DS}_r$ on randomly perturbed meshes.

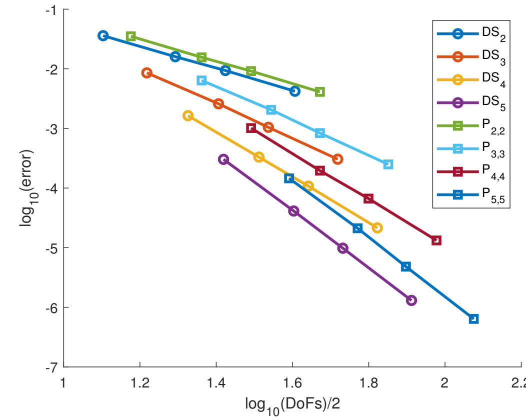
An example.



True solution



L^2-norm

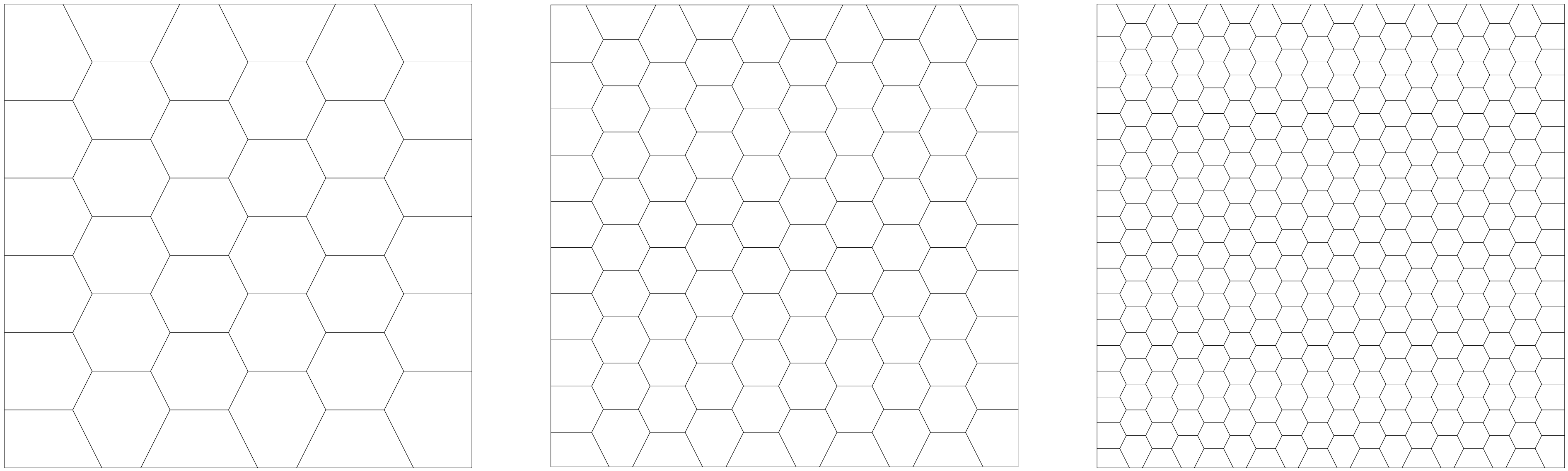


H^1-seminorm

True solution $u(x, y) = \left[1 + 0.1 \sin\left(\pi \cos(\pi x) \cos(3\pi y/2)\right)\right](1 - x)$, akin to the pressure in a linear flood shown on the left. Log scale L^2-norm (center) and H^1-seminorm (right) errors versus the number of DoFs for tensor product $\mathbb{P}_{r,r}$ and $\mathcal{DS}_r$ on trapezoidal meshes $\mathcal{T}_h^2$, $r = 2, 3, 4, 5$.

Remark. For this problem, $\mathcal{DS}_r$ outperforms $\mathbb{P}_{r,r}$ for all $r = 2, 3, 4, 5$.

Meshes. $N = 6$ mostly, but some $N = 4, 5$



36 elements   100 elements   324 elements

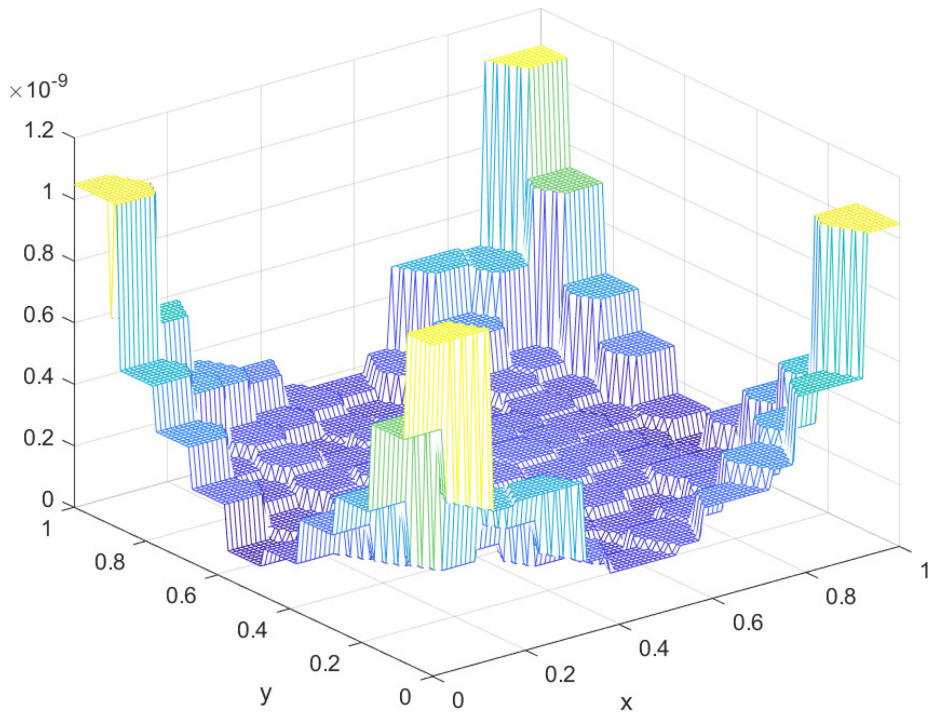
Meshes from PolyMesher (Talischi, Paulino, Pereira & Menezes 2012)

Errors and convergence rates for direct serendipity spaces

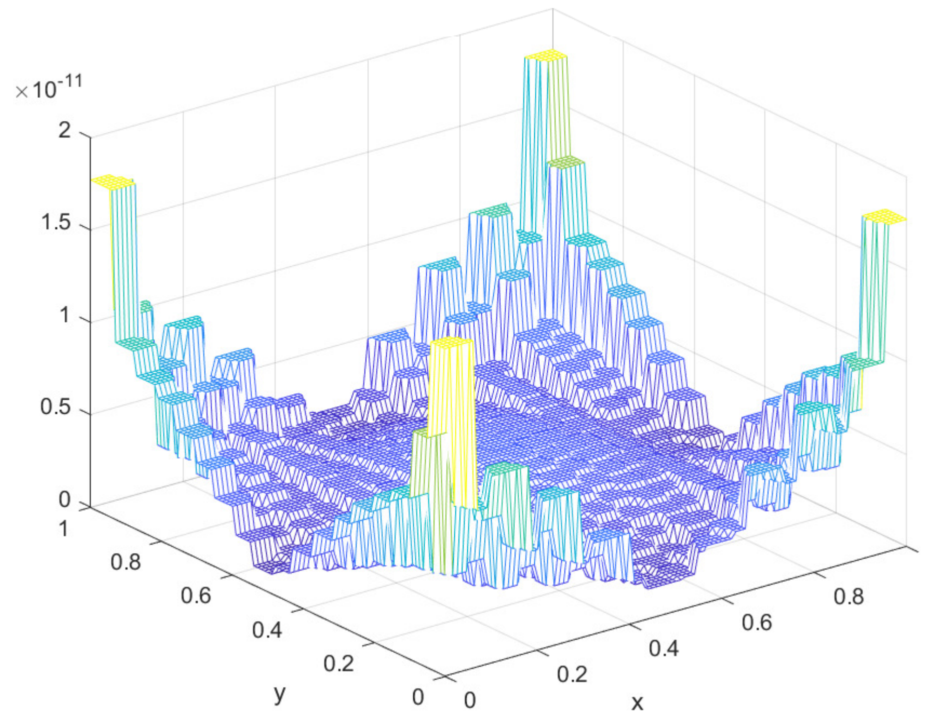
n	h	$r=2$		$r=3$		$r=4$		$r=5$	
					L^2-norm				
4	0.763	3.947e-02	—	1.461e-02	—	2.487e-03	—	3.882e-04	—
36	0.254	1.017e-03	3.32	7.804e-05	4.52	5.508e-06	5.37	2.740e-07	6.52
100	0.153	1.991e-04	3.16	8.639e0-6	4.25	3.549e-07	5.35	9.891e-09	6.48
196	0.109	6.960e-05	3.12	2.129e-06	4.14	5.921e-08	5.31	1.152e-09	6.33
324	0.085	3.199e-05	3.09	7.595e-07	4.09	1.568e-08	5.28	2.384e-10	6.24
			$\mathcal{O}(h^3)$		$\mathcal{O}(h^4)$		$\mathcal{O}(h^5)$		$\mathcal{O}(h^6)$
					H^1-seminorm				
4	0.763	1.066e-01	—	4.639e-02	—	1.032e-02	—	1.922e-03	—
36	0.254	9.801e-03	2.24	9.334e-04	3.36	8.123e-05	4.35	4.296e-06	5.50
100	0.153	3.223e-03	2.16	1.826e-04	3.15	8.844e-06	4.33	2.669e-07	5.41
196	0.109	1.575e-03	2.12	6.441e-05	3.08	2.083e-06	4.29	4.383e-08	5.36
324	0.085	9.285e-04	2.10	2.985e-05	3.05	7.138e-07	4.25	1.150e-08	5.32
			$\mathcal{O}(h^2)$		$\mathcal{O}(h^3)$		$\mathcal{O}(h^4)$		$\mathcal{O}(h^5)$

Observation. For the same number of elements, the error on this polygonal mesh is smaller compared to a mesh of trapezoids.

Suggestion. Elements with more edges might tend to give better approximations.

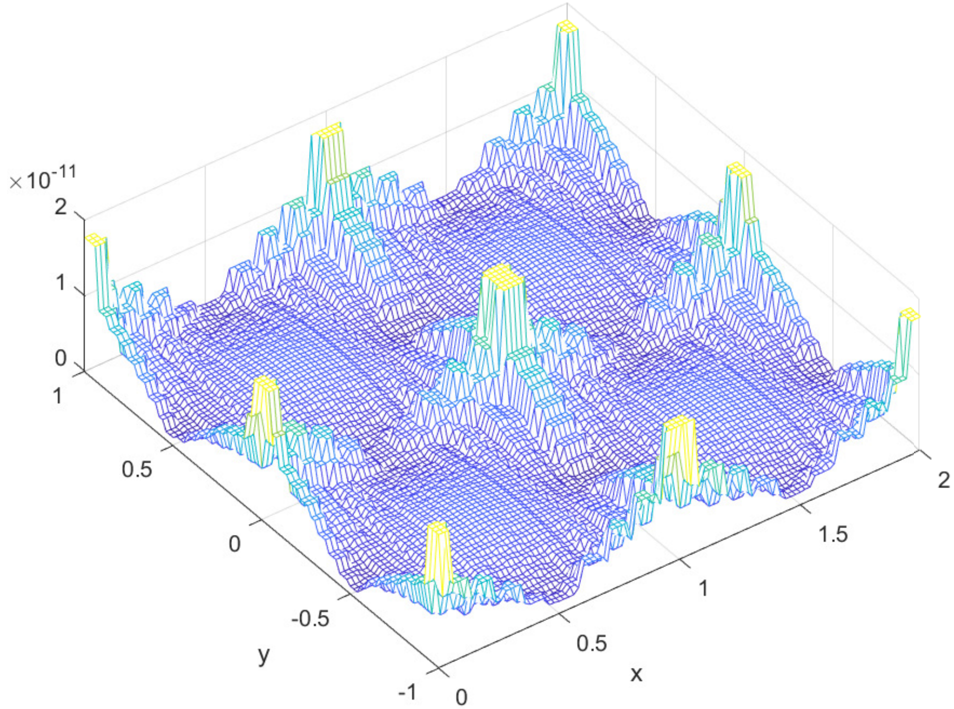


$n = 10$

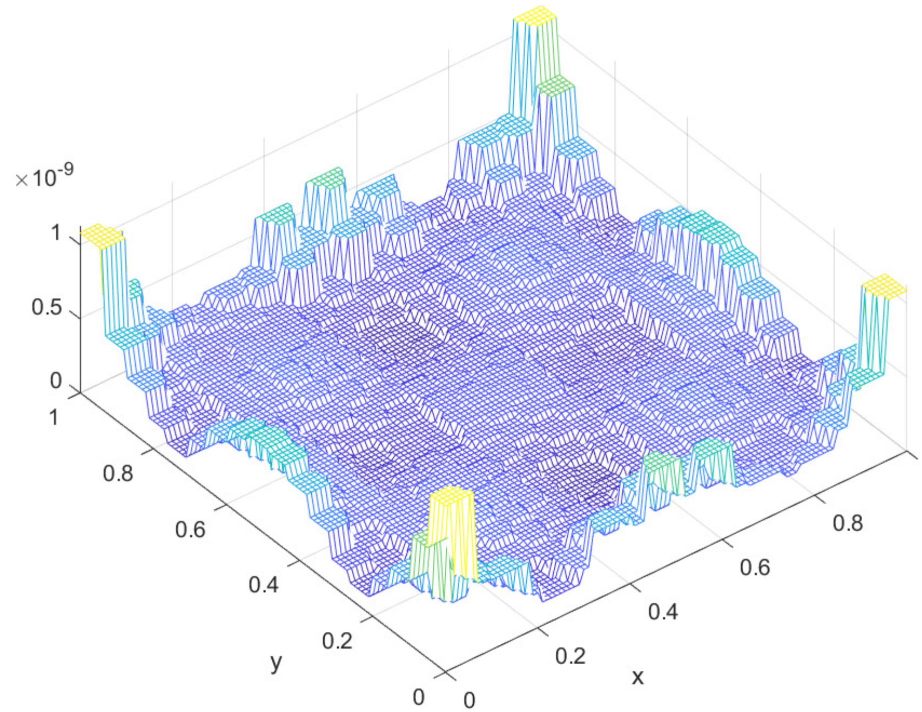


$n = 18$

The L^2 error on each element for mesh sequence $\mathcal{T}_h^1$ at level $n = 10$ and $n = 18$ with approximation index $r = 5$.



Larger domain



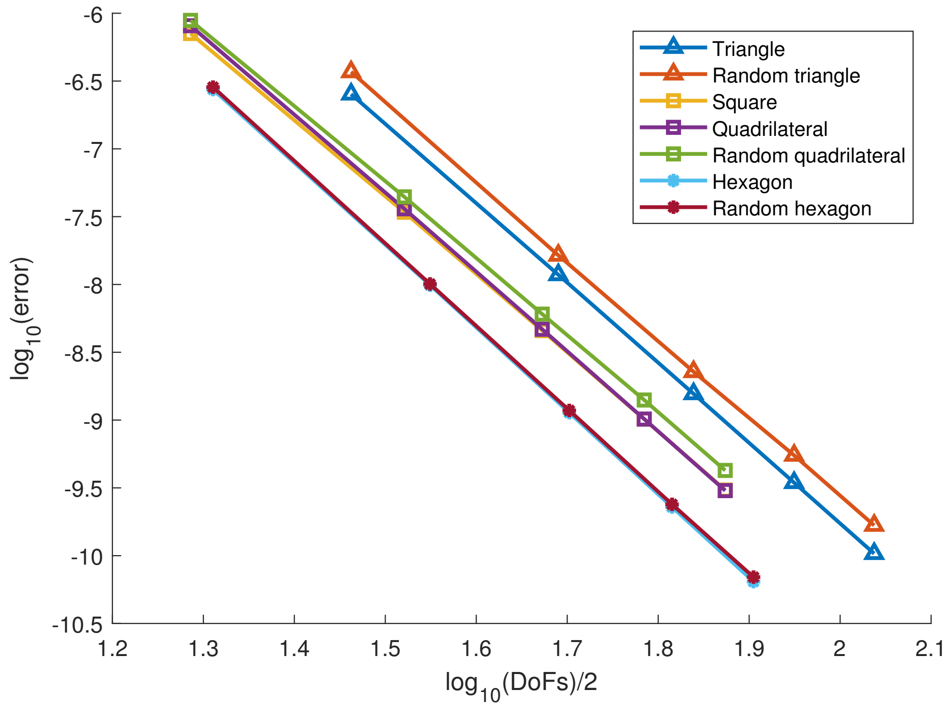
Exact solution with four humps

The L^2 error on each element for the two additional tests based on mesh sequence $\mathcal{T}_h^1$ at level $n = 18$ with approximation index $r = 5$.

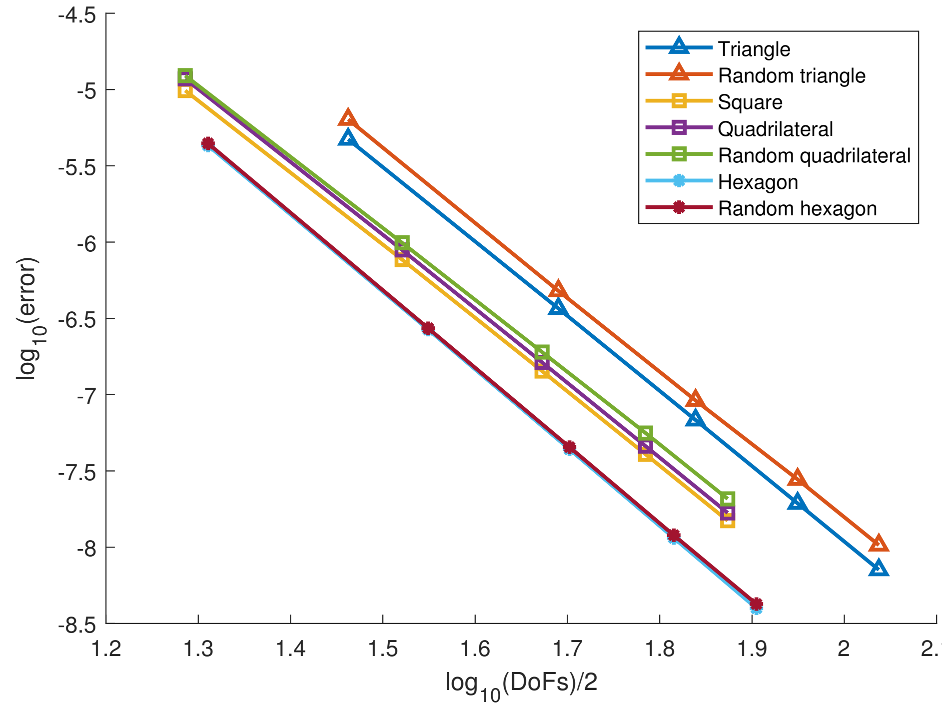
Remarks.

- When the original boundary elements are moved to the interior of the domain, we still observe the same larger error.
- Humps in the original manufactured solution do not affect the error as much as the shape of the elements.

Convergence Study for $\mathcal{DS}_r$ on Polygons – 5



L^2-norm



H^1-seminorm

Log of the L^2-norm and H^1-seminorm errors versus half the log of the number of DoFs on seven different mesh sequences with $n = 6, 10, 14, 18, 22$ and $r = 5$.

Convergence Study for $\mathbf{V}_r^s$ on Polygons

Errors and convergence rates for direct mixed spaces

n	h	$\|\|p - p_h\|\|$ error	rate	$\|\mathbf{u} - \mathbf{u}_h\|$ error	rate	$\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|$ error	rate
$r = s = 1$, full H(div)-approximation							
36	0.254	2.452e-02	2.05	8.125e-03	2.42	2.450e-02	2.04
100	0.153	8.641e-03	2.04	2.403e-03	2.37	8.639e-03	2.04
196	0.109	4.363e-03	2.03	1.094e-03	2.33	4.363e-03	2.03
324	0.085	2.624e-03	2.02	6.133e-04	2.29	2.624e-03	2.02
			$\mathcal{O}(h^2)$		$\mathcal{O}(h^2)$		$\mathcal{O}(h^2)$
$r = 1$, $s = 0$, reduced H(div)-approximation							
36	0.254	2.296e-01	1.22	5.183e-02	2.09	2.152e-01	1.04
100	0.153	1.308e-01	1.08	1.820e-02	2.04	1.277e-01	1.02
196	0.109	9.196e-02	1.04	9.199e-03	2.03	9.084e-02	1.01
324	0.085	7.104e-02	1.02	5.539e-03	2.02	7.051e-02	1.01
			$\mathcal{O}(h^1)$		$\mathcal{O}(h^2)$		$\mathcal{O}(h^1)$
$r = s = 2$, full H(div)-approximation							
36	0.254	1.853e-03	3.14	4.217e-04	3.37	1.853e-03	3.14
100	0.153	3.858e-04	3.06	7.535e-05	3.37	3.858e-04	3.06
196	0.109	1.385e-04	3.04	2.420e-05	3.37	1.385e-04	3.04
324	0.085	6.464e-05	3.03	1.038e-05	3.37	6.464e-05	3.03
			$\mathcal{O}(h^3)$		$\mathcal{O}(h^3)$		$\mathcal{O}(h^3)$
$r = 2$, $s = 1$, reduced H(div)-approximation							
36	0.254	2.452e-02	2.05	2.393e-03	3.08	2.450e-02	2.04
100	0.153	8.640e-03	2.04	5.053e-04	3.04	8.639e-03	2.04
196	0.109	4.363e-03	2.03	1.825e-04	3.03	4.363e-03	2.03
324	0.085	2.624e-03	2.02	8.545e-05	3.02	2.624e-03	2.02
			$\mathcal{O}(h^2)$		$\mathcal{O}(h^3)$		$\mathcal{O}(h^2)$

4. Summary and Conclusions

1. Conforming finite elements on polygons are important in many areas.
   - Solving PDEs in certain applications.
   - General interpolation and approximation of functions.
   - Visualization.
2. Direct spaces (polynomials plus supplements) offer advantages
   - Quadrilaterals: no accuracy loss due to reference element mapping.
   - Polygons: require no reference element.
3. Direct serendipity finite elements developed for convex polygons.
   - H^1-conforming and fully constructive.
   - Minimal DoFs and approximate optimally on shape regular meshes.
4. Direct mixed finite elements developed for convex polygons.
   - $H(\text{div})$-conforming direct and fully constructive.
   - Arise from a de Rham complex using FEEC.
   - Full and reduced $H(\text{div})$-approximating spaces.
   - Minimal DoFs and approximate optimally on shape regular meshes.

5. Numerical results compare DoF efficiency.
   - On quadrilaterals, whether mapped $\mathbb{P}_{r,r}$ or $\mathcal{DS}_r$ has higher DoF efficiency depend highly on the true solution.
   - Mesh elements with more edges give better approximation.
6. Extensions to 3-D polytopes.
   - We have defined some mixed spaces on hexahedra.
   - Direct Serendipity spaces are difficult because:
     - Edge and face basis functions both need supplements;
     - There is a much wider range of possible shapes;
     - Some polytopes have no nonadjacent faces (e.g., pyramids).
   - Direct Mixed spaces are difficult because:
     - The de Rham complex is longer

$$\mathbb{R} \hookrightarrow H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0$$

     - We need to define a space of vectors $\mathbf{X}$ so that

$$\mathbb{R} \hookrightarrow \mathcal{DS}_{r+2}(E) \xrightarrow{\text{grad}} \mathbf{W}_{r+1}(E) = \text{grad}\,\mathcal{DS}_{r+2}(E) \oplus \mathbf{X} \xrightarrow{\text{curl}} \mathbf{V}_r^s(E) = \text{curl}\,\mathbf{X} \oplus \mathbf{x}\mathbb{P}_s(E) \xrightarrow{\text{div}} \mathbb{P}_s(E) \longrightarrow 0$$

References

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Let $\mathcal{T}_h$ be a finite element partition of the domain into convex quadrilaterals.

Shape regularity assumption. For $E \in \mathcal{T}_h$

$$h_E = \text{diameter of } E$$
$$\rho_E = 2\min\{\text{diameter of largest circle inscribed in sub-triangle of } E\}$$

A collection of meshes $\{\mathcal{T}_h\}_{h>0}$ is uniformly shape regular if there exists $\sigma_* > 0$ such that

$$\frac{\rho_E}{h_E} \geq \sigma_* > 0, \quad \forall E \in \mathcal{T}_h$$

Remark. A shape regular mesh has a finite range of possible edges an element could have. It also has a bound on the number of elements that can share a single vertex.

We can use a compactness argument to handle the geometry.