

NUMERICAL STUDY: FLUX THROUGH A FRACTAL ROBIN BOUNDARY

DOUGLAS N. ARNOLD, CHUNING WANG, QILE YAN

UNIVERSITY OF MINNESOTA



INTRODUCTION

• Problem formulation:

- Domain is a square with a fractal top
- Laplace's equation
- Dirichlet at bottom, Neumann on sides
- Robin boundary conditions on the fractal edge

$$\begin{aligned} & \Lambda \frac{\partial u}{\partial n} + u = 0 \\ & \ell = 1/3^2 \quad L_p = 4^2/3^2 \\ & -\operatorname{div}(D \operatorname{grad} u) = 0 \\ & \frac{\partial u}{\partial n} = 0 \quad u = 1 \quad \frac{\partial u}{\partial n} = 0 \end{aligned}$$

Problem parameters

ℓ : length of shortest segment in pre-fractal

L_p length of pre-fractal edge

Λ : Robin parameter (as Λ varies from 0 to $+\infty$, the bc varies from Dirichlet to Neumann)

Total flux through the top, reciprocal flux

$$\begin{aligned} \Phi(\Lambda) &:= \int_{\text{top}} -D \frac{\partial u}{\partial n} d\sigma = \frac{D}{\Lambda} \int_{\text{top}} u d\sigma \in \mathbb{R} \\ Z(\Lambda) &:= \frac{1}{\Phi(\Lambda)} - \frac{1}{\Phi(0)} \end{aligned}$$

• Study Goal:

- Question: How does the flux $\Phi(\Lambda)$ vary with Λ ?
- Answer: $Z(\Lambda)$ depends on Λ with clear patterns in three regimes

1. Near Dirichlet: $\Lambda \ll \ell$
2. Transition stage: $\ell < \Lambda < L_p$
3. Near Neumann: $\Lambda \gg L_p$

NUMERICAL SOLUTION AND EXPECTATIONS

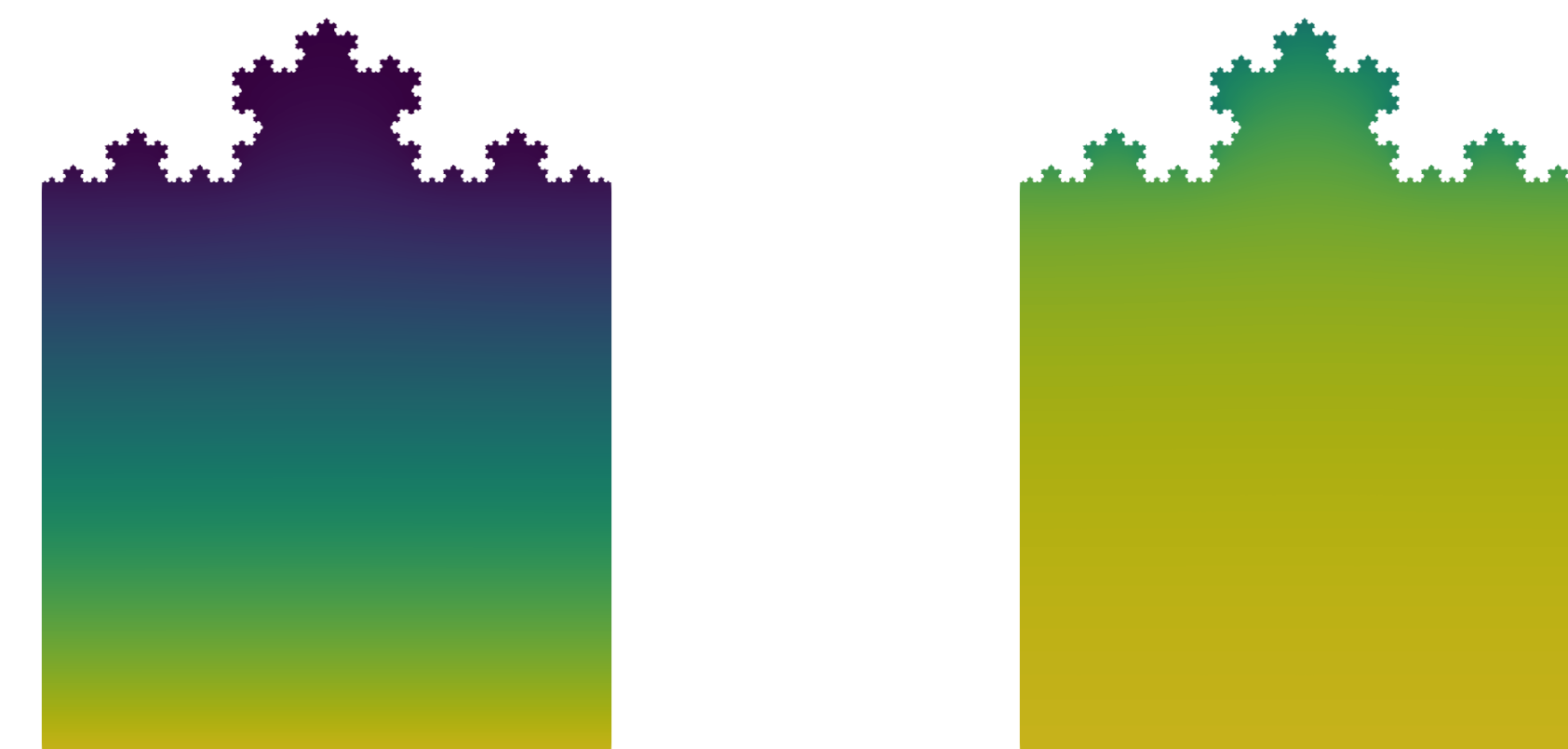
Linear v.s. log-scaled color map



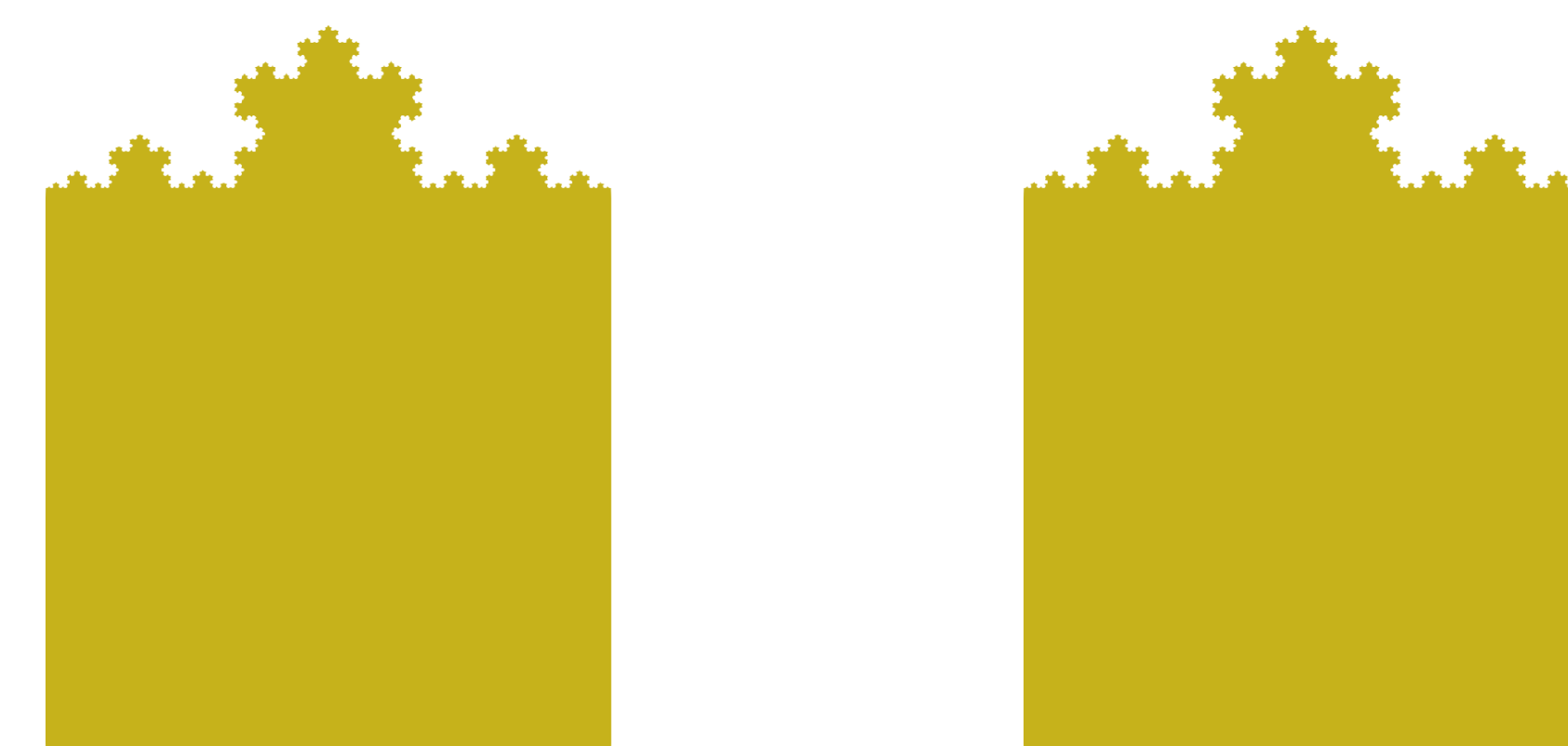
- $\Lambda = \ell \times 10^{-3}$: the 1st regime, near Dirichlet.



- $\Lambda = (\ell L_p)^{1/2}$: the 2nd regime, intermediate stage.



- $\Lambda = L_p \times 10^3$: the 3rd regime, near Neumann.



• Expectations:

1. For each regime i ($i = 1, 2, 3$), $\exists \alpha_i, \beta_i$, s.t.

$$Z(\Lambda) \approx \alpha_i \Lambda^{\beta_i}.$$

2. We expect

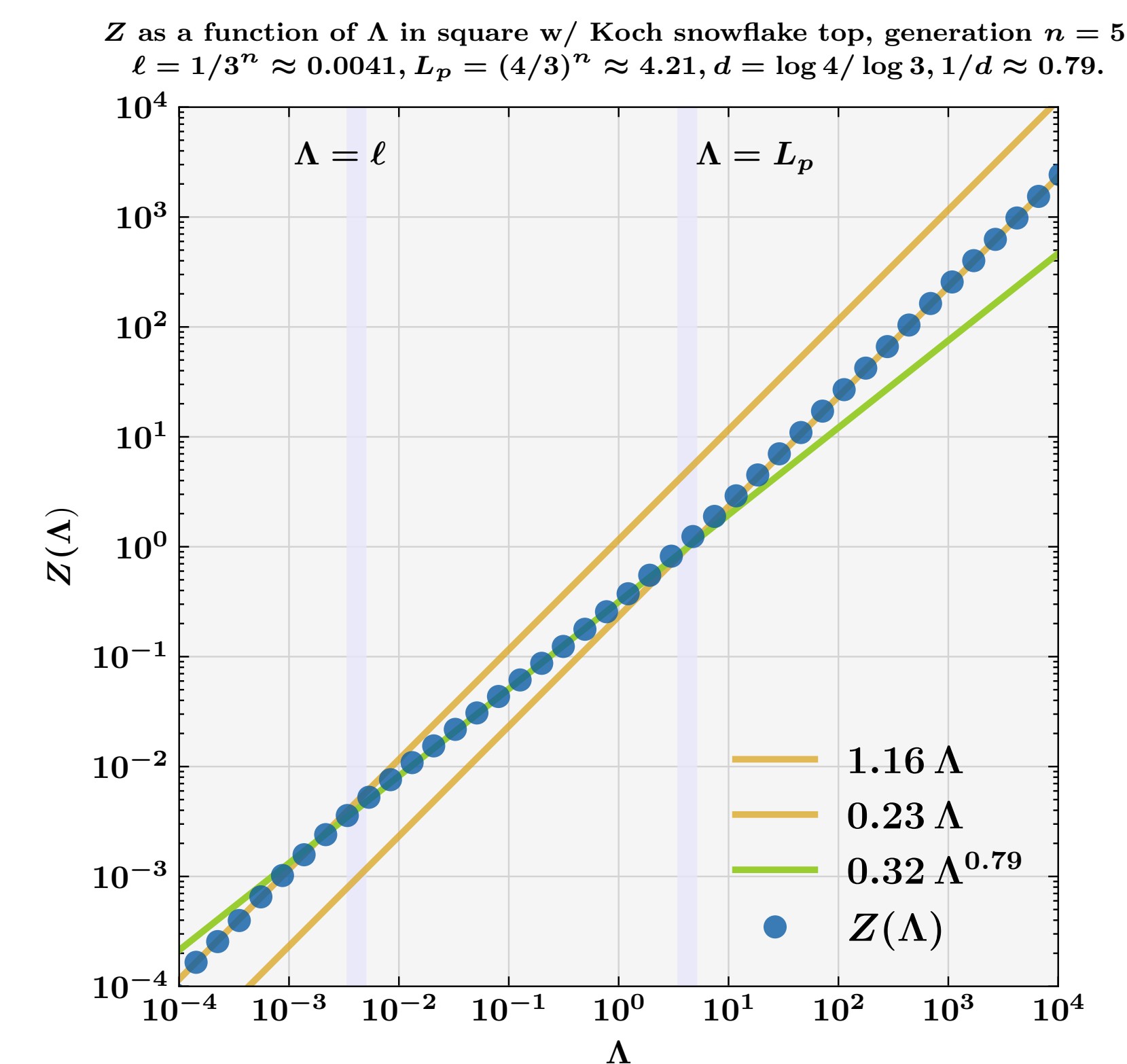
$$\beta_1 \approx 1, \quad \beta_2 \approx 1/\log_3 4 \approx 0.79, \quad \beta_3 \approx 1,$$

where $1/\log_3 4$ is the reciprocal of the Hausdorff dimension of the fractal structure.

3. As $\Lambda \rightarrow \infty$, we expect

$$\Phi \rightarrow DL_p/\Lambda \Rightarrow \alpha_3 \approx 1/DL_p \approx 0.24.$$

Study $Z(\Lambda)$ vs. Λ in three regimes



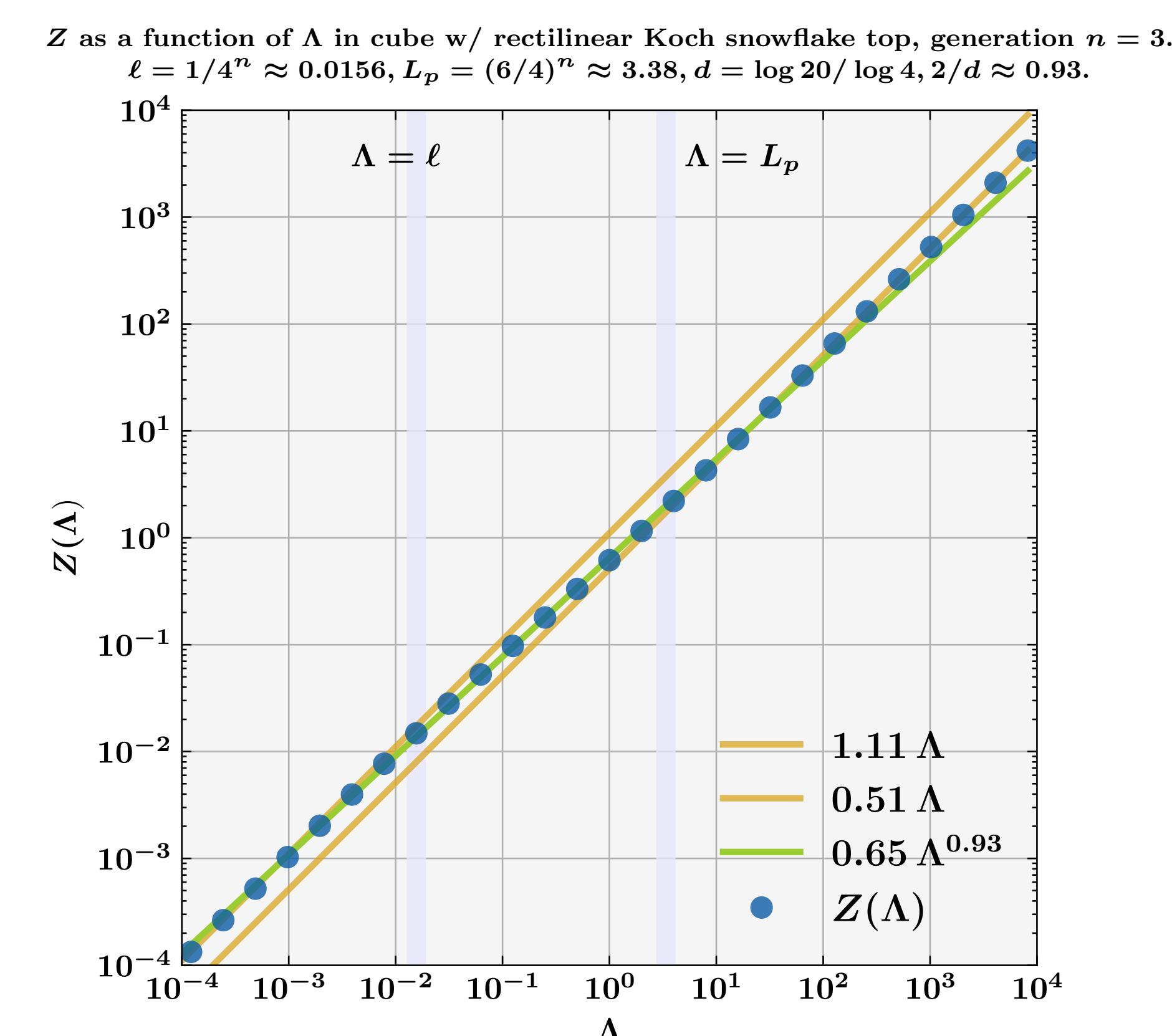
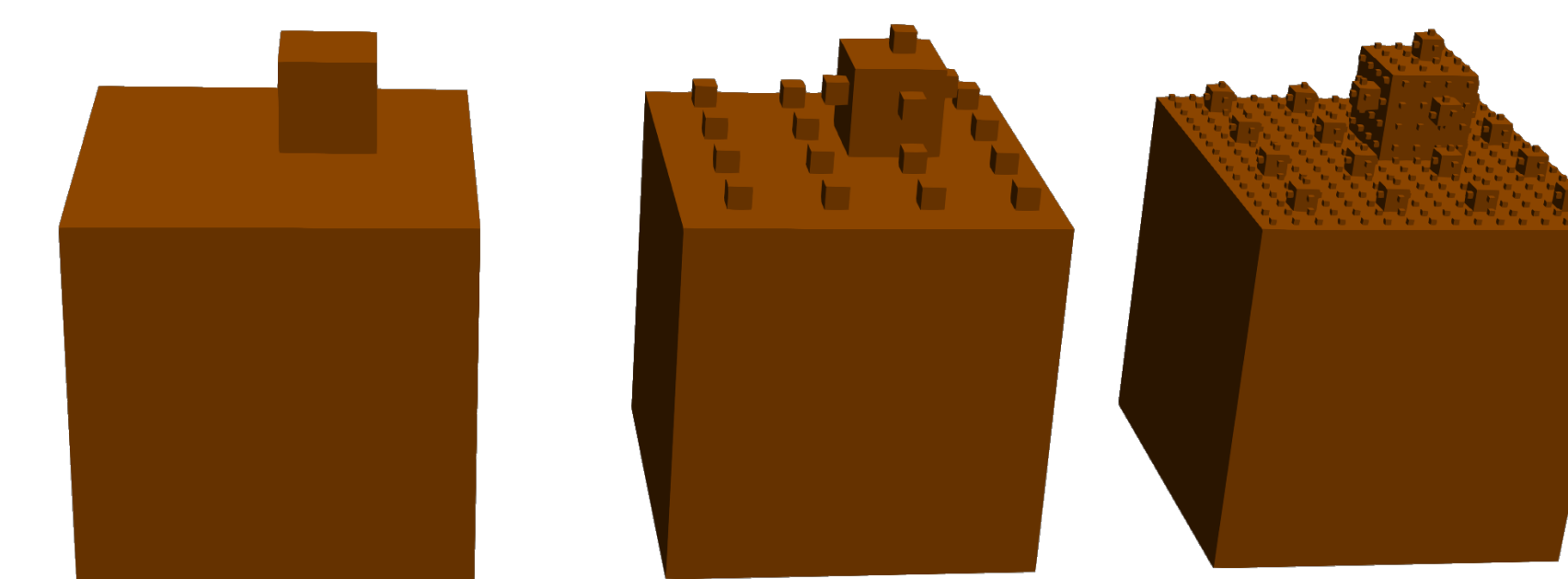
- Study β_i with different pre-fractal level n .

	n		
	2	4	5
β_1	0.97	0.94	0.95
β_2	0.87	0.81	0.79
β_3	1.00	1.00	1.00

As $n \uparrow 5$,

$$\beta_2 \downarrow 1/\log_3 4$$

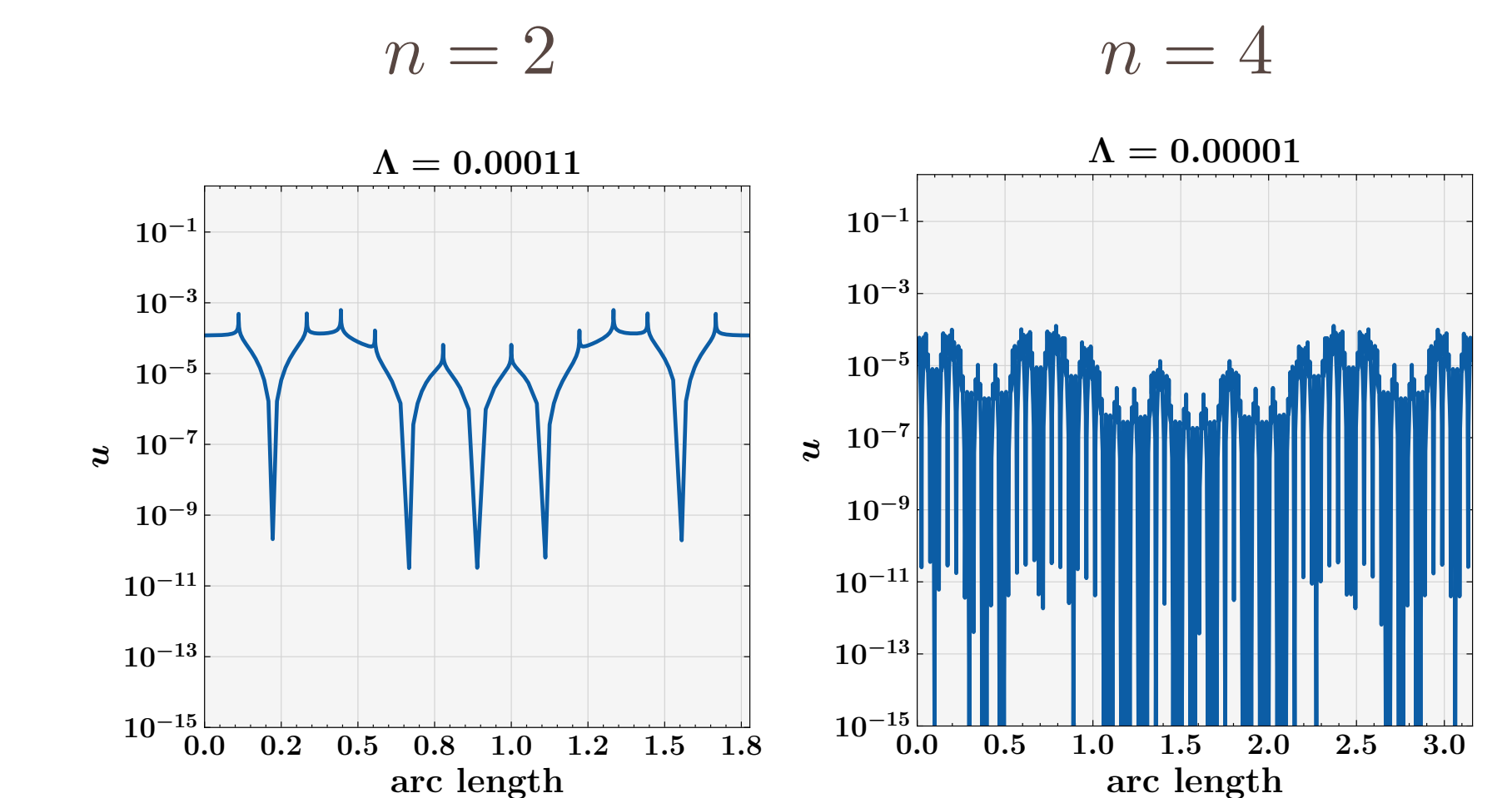
Generalization to 3D



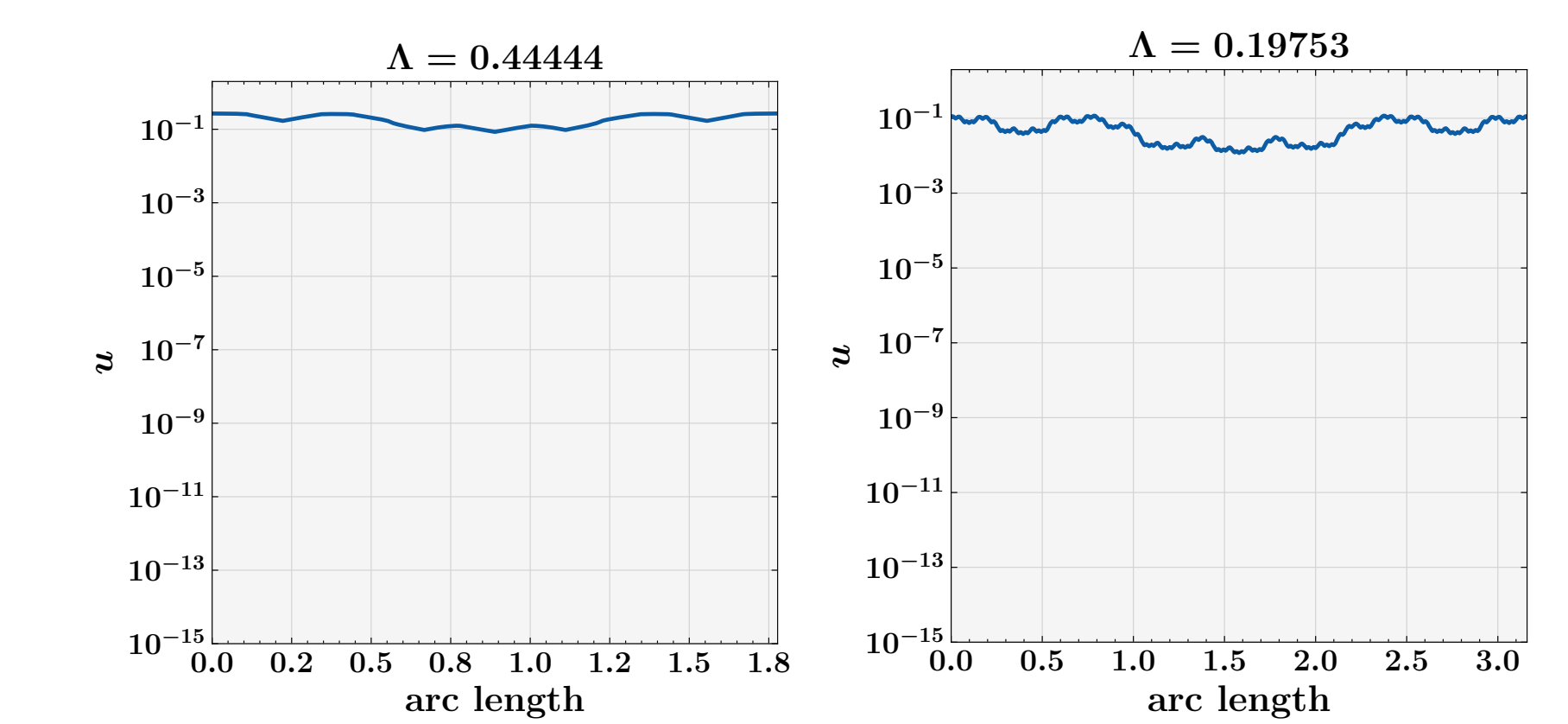
In 3D, $\beta_2 \approx 2/\log_4 20 = 0.93$, twice the reciprocal of the Hausdorff dimension of the fractal!

OTHER QUANTITIES OF INTEREST

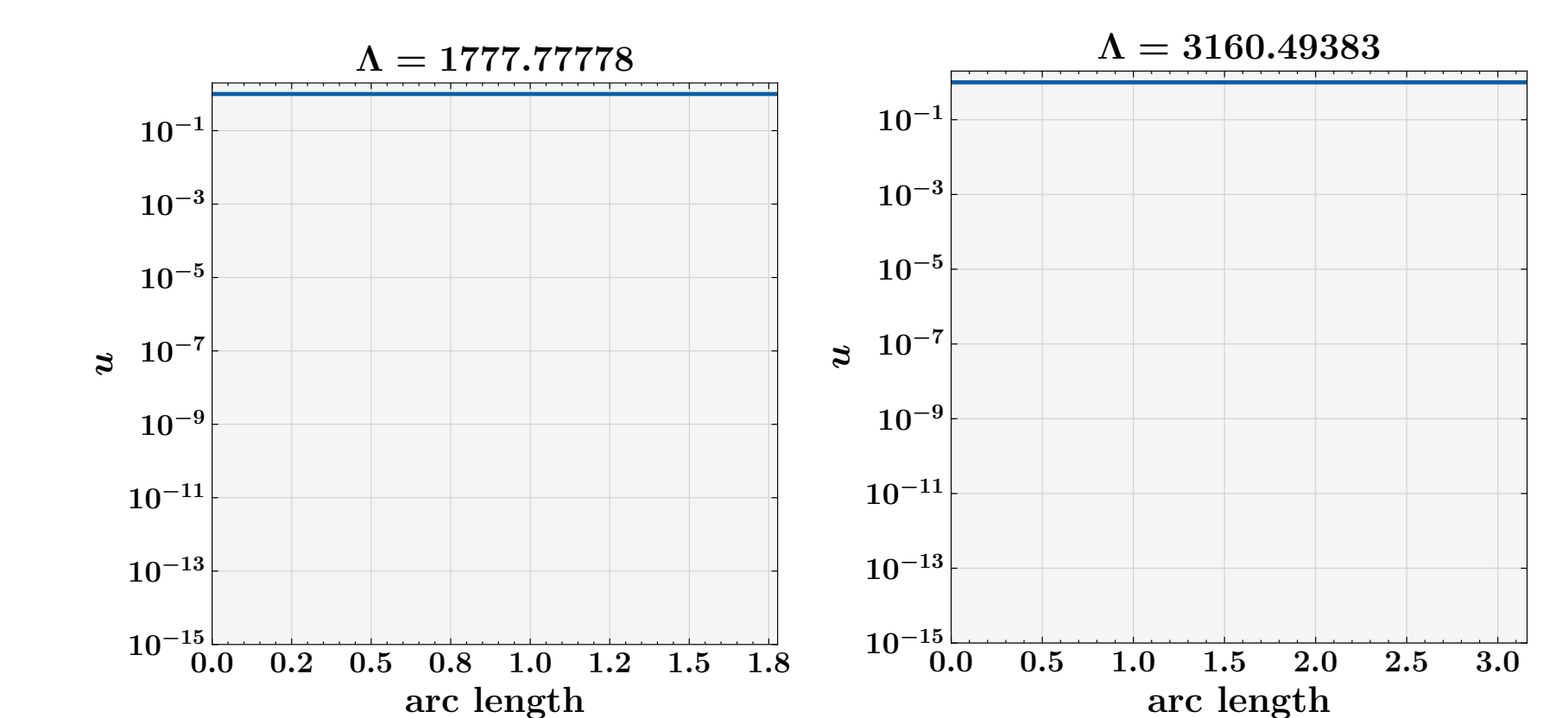
u (log-scaled) vs. the arclength on top



$\Lambda = \ell \times 10^{-3}$: the 1st regime, fluctuation near 0.

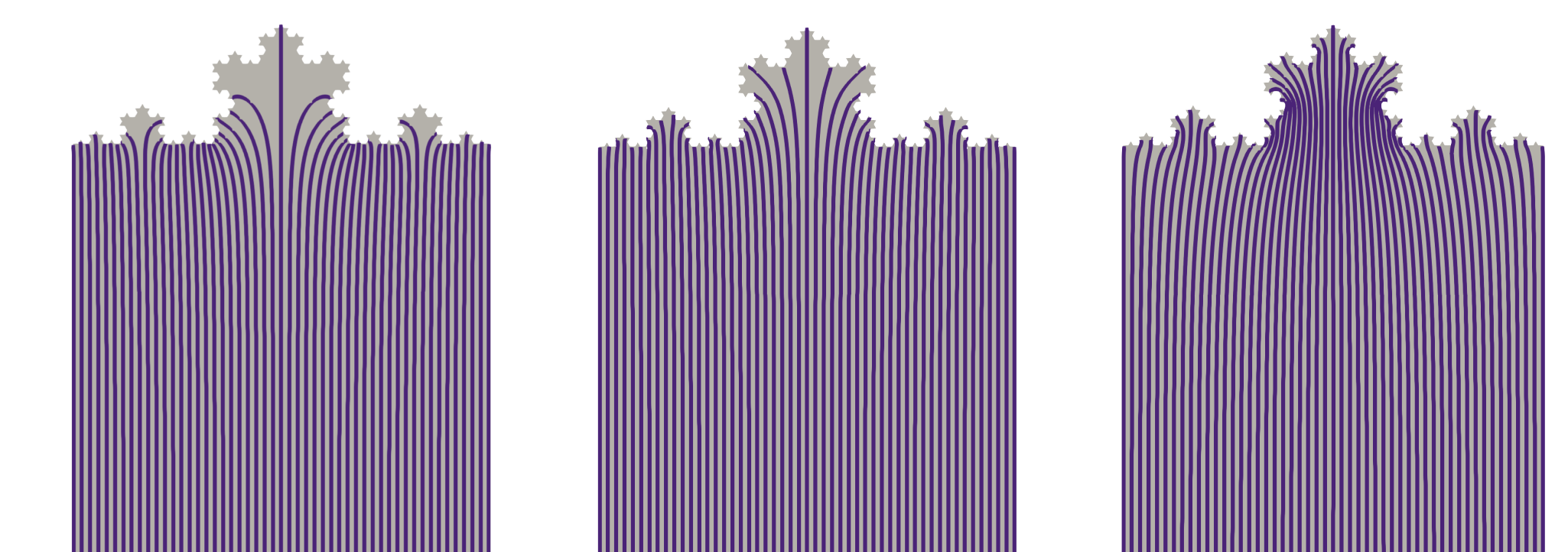


$\Lambda = (\ell L_p)^{1/2}$: the 2nd regime, arise from 0 to 1.



$\Lambda = L_p \times 10^3$: the 3rd regime, converge rapidly to 1.

The streamline plots of ∇u



$n = 4$, $\Lambda = \ell \times 10^{-3}$, $(\ell L_p)^{1/2}$, $L_p \times 10^3$.

The direction of ∇u is $\sim (0, -1)$ in most part, varying rapidly only near the fractal top.

REFERENCES

Filoché, M., and Sapoval, B., *A simple method to compute the response of non-homogeneous and irregular interfaces: electrodes and membranes*. Journal de Physique I, 7 no. 11 (1997): 1487-1498.